Solid Mechanics and Its Applications

Volume 194

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**Aims and Scope of the Series**

The fundamental questions arising in mechanics are: *Why?*, *How?*, and *How much?* The aim of this series is to provide lucid accounts written by authoritative researchers giving vision and insight in answering these questions on the subject of mechanics as it relates to solids.

The scope of the series covers the entire spectrum of solid mechanics. Thus it includes the foundation of mechanics; variational formulations; computational mechanics; statics, kinematics and dynamics of rigid and elastic bodies; vibrations of solids and structures; dynamical systems and chaos; the theories of elasticity, plasticity and viscoelasticity; composite materials; rods, beams, shells and membranes; structural control and stability; soils, rocks and geomechanics; fracture; tribology; experimental mechanics; biomechanics and machine design.

The median level of presentation is the first year graduate student. Some texts are monographs defining the current state of the field; others are accessible to final year undergraduates; but essentially the emphasis is on readability and clarity.
Preface

Why This Book?

It is my hope that the present book is the final edition of my lecture notes Elementary Continuum Mechanics for Everyone. The very first edition was written in connection with my teaching a first course on continuum mechanics at the Technical University of Denmark. I tried to find a text that would suit my intentions, but found that the books on the market either were filled with tensor analysis, including e.g. curvilinear coordinates and Christoffel Symbols, or they were of the old mechanics traditions, meaning that every new example was treated separately with the result that universally valid principles, such as the Principle of Virtual Work, never appeared. In my opinion, if teaching continuum mechanics at any level is justified it must contain a strong element of general statements. Then, of course, there is the risk that the treatment becomes mathematically so difficult that it cannot serve as an introduction to the subject. Therefore, this book contains an elementary, but quite general, exposition of the subject, and it is my sincere expectation that most students—with some effort, of course—should be able to get a feel for the important concepts of (generalized) strains, (generalized) stresses, and the Principle of Virtual Work. Judging from the experience of many of my predecessors I may have set too high a goal, but still I shall try to reach it.

I feel that a solid understanding of almost any subject may only be obtained through examples, and this is one of the reasons why I have included Parts II–V, which may be viewed as applications of the theory of Part I. Of course, I consider the topics of Parts II–V to be important by themselves, but in the present context it is just as pertinent that they may be based on and illustrate concepts from Part I. Thus, an essential idea behind this book is to present continuum mechanics, not only as a valuable subject in its own right, but as the foundation for all our theories and methods governing the behavior of solids and structures. Although I endorse the use of examples it has been my experience that they tend to obscure the real subject by their sheer number and length. Therefore, I have limited the number of worked examples, except in Part III because there the examples provide useful information about properties of a number of cross-sections. I hope that I have found a reasonable balance in this regard.

Later, we shall see that there are more than just one principle of virtual work, but at this point this is not important.
Who Should Read this Book?

The present book is meant as an introduction to continuum mechanics with applications in solid and structural mechanics and, obviously, engineering students are potential readers. It is, however, my hope that engineers who would like to achieve a better understanding of the theories they apply will appreciate this book.

Other Possible Topics

As regards topics not treated here there is a couple that I would have liked very much to include. An important one is Elastic Fracture Mechanics. The reason why I have left it out is that in order to do the job right one has to introduce solutions to some fundamental elasticity problems, which can almost only be found by use of complex functional analysis. If I were to introduce the methods of Muskhelishvili (1963) that particular subject would have taken up much more space than I consider feasible in the present context. The possibility of presenting the solutions without proof and then utilizing them as a basis for the theory of linear fracture mechanics may be justified under other circumstances, but not here, where the emphasis is different, namely on full derivations of all\textsuperscript{2} formulas.

Another subject which might have been covered is perfect plasticity because results from computations by hand based on this assumption have been used over and over during the past sixty years or so. However, with few exceptions, computer analyses that consider the more realistic case of strain hardening have been made possible due to the small cost of personal computers and have therefore made the older methods redundant. In my opinion, today the most important application of perfect plasticity is Johansen’s Yield Line Theory\textsuperscript{3}, see e.g. (Johansen 1963) and (Johansen 1972). The beauty of his theory is that it may provide useful results with only a limited effort, even for rather complicated situations. On the other hand, derivation of the Yield Line Theory would take up more space than I consider reasonable in the present context.

My Writing Style

Readers of this book will soon discover that its style is somewhat unusual—probably an older reader will find it appalling, see also the Introduction, but I have my reasons for writing the way I do. I have tried to write in an informal style without sacrificing accuracy. A main idea behind my

\textsuperscript{2} There is one exception to this, namely the formulas for imperfection sensitivity in Chapter 20. The reason for this is that the formulas by themselves are short, while their derivation is long and very complicated, see e.g. (Budiansky 1974) or (Christensen & Byskov 2010).

\textsuperscript{3} Johansen himself never acknowledged that his yield line theory actually could be viewed as an example of upper bound solutions of the more general theory of perfect plasticity.
writing is that I attempt not to introduce any concept without first giving a motivation for its usefulness and relevance. I may not have succeeded, but this was my goal.

Many years ago when I was an undergraduate at the Technical University of Denmark I encountered the subject of mechanics in a physics course. It was never explained to us why the subject as a whole was important, and in particular there never was an indication of the reason behind the way the different topics were dealt with—other than the comment: “If you don’t understand this, you don’t belong here.” In my case they may have been right, but at that time I did not feel that a statement like that showed much insight into how the human mind works, and today I think that teachers like that should not have been allowed to stand in front of a class. I intend to avoid such arrogance, but caution that the consequence in some cases is that the presentation will seem unnecessarily lengthy to some readers.

You might say that my ambition has been to explain “why—not just how.”

In all probability it is not necessary to mention that English is not my mother tongue, but hopefully the meaning of my efforts is clear in spite of that.

If, on the other hand, the language of this book bears some resemblance to (American) English, my wonderful secretary at Aalborg University, Kirsten Aakjær, who has read all the pages\textsuperscript{P.4} of a shorter, previous edition and has explained to me why the content was different, deserves to be mentioned. Kirsten Aakjær favors British English, and many times she tried to convert every phrase into that language. Over the years, we have had discussions about the virtues of these two kinds of English, but have never obtained a complete agreement—only a friendly ceasefire.\textsuperscript{P.5}

It is also worth mentioning that Kirsten Aakjær in many cases suggested more fundamental improvements to my writing, such as telling me that a particular example had a too abrupt ending.

Being a very stubborn person, I have sometimes chosen to follow my own instincts in spite of the advice by Kirsten Aakjær, so blame me—not her—for the errors.

As you will see later, in order that you may have a quick view of the topics I have put a lot of gray boxes in the margin.

\textsuperscript{P.4} It must have been a boring job considering the fact that the contents must have been alien to Kirsten Aakjær.

\textsuperscript{P.5} In the mid-Seventies my family and I stayed in Massachusetts on a sabbatical leave at Harvard University. My son, Torben, who was five or six years old at that time, once proclaimed: “British English is bad English,” which is a statement that I have often quoted since.

To be fair, our year in Massachusetts was the Bicentennial year, and the British were always the bad guys when the kindergarten kids played.

\textsuperscript{P.4} Many gray boxes in margin
Open Source Programs
I could not have written this book without the use of open source and free programs such as \LaTeX\, \LaTeX \textsc{2}, \LaTeX \textsc{2e}, \LaTeX \textsc{2ε}, Bib\LaTeX, makeindex, texindy,\textsuperscript{P.6} gnuplot, maxima, Octave, xfig, gcc, and g++. 

Possible Errors in This Book
I am sure that there must be some errors even in this edition. There is, unfortunately, nobody else but myself to blame: I have pressed the keys on the keyboard, I have moved and clicked the mouse, I have written the text, and I have drawn all figures and produced all the plots.

Esben Byskov
Jægersborg and Aalborg
August 14, 2012

\textsuperscript{P.6} Just the thought of using one of the WYSIWIG, sometimes called WYSIAIG for “What You See Is All You Get,” text editing programs brings sweat to my face.
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Introduction

What Is Continuum Mechanics?

Quite trivially, continuum mechanics per se deals with the description of deformations of three-dimensional continua, i.e. models whose properties are independent of scale in that the continuum does not possess a structure.\(^1\)\(^7\) Thus, continuum mechanics does not try to model the atomic structure of the involved materials—perhaps not even the crystalline, or spongy, or lumpy structure—but offers a “smeared-out” version of the real world. Also, the desired description depends very much on the needs of the discipline in question. In solid mechanics—which is the sub-discipline of continuum mechanics that I treat in this book—material elements that originally were neighbors remain neighbors throughout the lifetime of the material, with the possible exception of a crack forming and separating the neighbors. In fluid mechanics, on the other hand, it is necessary to be able to handle cases where neighboring material elements cease to be neighbors because they have been separated by other material elements that have moved in between them.

Thus, continuum mechanics is many different things depending on the purpose. In the following chapters I give an introductory exposition of the continuum mechanics of solids.

Many readers—in particular older ones—may feel that the emphasis is completely wrong and that subjects which are left out here are much more important than the ones included.

Also, some may think that in order to motivate the students I should have shown lots of results from computer analyses in the form of plots of strains and stresses\(^1\)\(^8\) and from experiments. I refuse to believe that the students of today are incapable of understanding the importance and usefulness of a subject unless it is presented in terms of cartoons.\(^1\)\(^9\) Furthermore, by showing such results in terms of colored plots the students might get the impression that in order to do an analysis you only have to get some commercial computer program, probably a Finite Element Program, and that

---

\(^1\)\(^7\) This does not preclude continuum theories that entail a length scale which is derived from analysis of materials that do have a structure, e.g. wood, fiber reinforced epoxy, and concrete. Such theories, which are more advanced than the theories described in this book, have become quite popular in recent years for analysis of material instabilities, such as “kinkbands” in wood and in fiber reinforced epoxy.

\(^1\)\(^8\) These terms will be introduced later, see Part I, Chapter 2.

\(^1\)\(^9\) Although I am a cartoon buff myself, I hope that I am correct in this belief.
therefore you do not have to know any theory. Nothing can be more wrong, as any experienced engineer or researcher knows—many mistakes in “the real world” are made by people who know too little about the theoretical background of the method they are using.

Be that as it may, I have made some choices that I feel are proper—knowing that a lot of useful subjects will not be covered. But, and this is my strong opinion, in an introductory course on continuum mechanics it is important that the students get a feel for the nature of strains and stresses and learn the central role played by the Principle of Virtual Work rather than being able to juggle Curvilinear Coordinate Systems, Co- and Contravariant Tensors, Christoffel Symbols, Mohr’s Circles, etc.

The reader will observe that the same topic is introduced several times. The reason for this is that in a number of cases I considered it easier for the reader to become acquainted with some new concept via an application or an example rather than through general statements. However, I am certain that it should not be too difficult to find the place where the more concise statements or formulas are given.

The most central obligation of continuum mechanics of solids is to provide a common basis, including a set of rather strict requirements, for all specialized theories, i.e. theories for bars, beams, plates, shells etc.\(^1\)

“\textit{I Need Continuum Mechanics Like I Need Another Hole in My Head}”

\textbf{“What’s wrong with P over A?”}

The way the curriculum at the Department of Building Technology and Structural Engineering, Aalborg University, is set up—and to my experience, it is very much like many other curricula—the students have been acquainted with analysis of bars and beams, and possibly plates, as well as structures made up of such elements before they meet the more general continuum mechanics. So, a student might say “I need continuum mechanics like I need another hole in my head now that I have already learned to handle the most important structural elements as well as structures made up of such elements.” That some professors share that way of thinking was told to me by one of my very good American friends\(^1\) who said that at a meeting at his university\(^1\) an older professor exclaimed “What’s wrong with P over A?” when the subject of continuum mechanics including strain

\footnotesize{\(^{1,10}\) Again, some may disagree with me on this.\(^^{1,11}\) I am very well aware that continuum mechanics in itself is a vast subject that has its own justification, but here I am thinking about our particular purpose of giving an introductory course.\(^^{1,12}\) He shall remain nameless in this connection.\(^^{1,13}\) The university shall also remain nameless.}
and stress tensors was mentioned.

So why bother spending time on learning continuum mechanics? The answer to this contains at least two ingredients. First, it may not be all that obvious to the students that there is a common foundation for the theories for bars, beams, etc. Second, there are a lot of other important cases that have not been covered by the theories already learned, and if the student encounters such theories later it is imperative that he or she knows the foundation for any sound continuum mechanical theory, including the specialized ones. Otherwise, the consequences may be dire.

The Main Emphasis of this Book

The main technical emphasis of this book is on general principles, most notably the Principle of Virtual Displacements\textsuperscript{1,14} that govern deformation of structures and structural elements. I consider it just as important to explain why we are interested in a particular subject and even more so not just how, but rather why we establish our theories the way we do.

We may consider the three-dimensional continuum the only “real” model of our world, but at the same time we must acknowledge that in many cases more primitive models may be more feasible in terms of computational effort and clearness of results. On the other hand, establishing equations for specialized continua, such as beams and plates, on an \textit{ad hoc} basis does not appeal to me for several reasons. Probably the most important is that you must be extremely careful—or lucky—to insure that such a theory obeys the general principles. The main stumbling block consists in the problem of ensuring that the stress and strain quantities are \textit{generalized}, i.e. are work conjugate, see Section 2.4.2 and Chapter 33. If they are not, the fundamental principles, such as the Principle of Minimum of the Potential Energy and the upper and lower bounds of the theory of perfect plasticity—a fairly important subject\textsuperscript{1,15} which is not dealt with in this book—do not apply. It is my sincere hope that after reading this book—and having done some work yourself—you will appreciate the importance of this.

There is, however, one major disadvantage of this approach, namely that the generalized quantities, usually the generalized stresses, sometimes are difficult to interpret in a physically obvious way, but, as you will see, we succeed as regards most beams, see Section 7.3, and plates, see Section 9.1, but encounter problems with some other beams, see Section 7.4.

\textsuperscript{1,14}For elastic structures I use the Principle of Minimum of the Potential Energy, see Chapter 33, more frequently, in part because I think that in some connections this provides a more clear exposition, and in part because some other principles and methods, e.g. the Rayleigh-Ritz Procedure, Section 18.3.8, are based on this.

\textsuperscript{1,15}Over the years, calculations by hand based on application of the theory of perfect plasticity have provided extremely useful solutions to engineering problems, but, to be fair, the theory makes very strong assumptions as regards the behavior of the materials, and today it is less important because of the advent of computers.
How to Read this Book

I strongly suggest that you start reading Part I, which deals with the “real” subject of this book.\textsuperscript{1.16} Then, if there is some mathematical subject or tool that you need you might try to find it in Part VI, although I cannot guarantee that it is covered there. The Index at the end of the book should be of help in this regard.

You may not appreciate the book until you have been through Parts II–VI,\textsuperscript{1.17} so I recommend that you do not stop after Part I.

Finally, I urge you to read the footnotes.\textsuperscript{1.18}

Expected Prerequisites

A reader with a working knowledge of calculus, linear algebra, and a handle on the concepts of rational mechanics, i.e. of forces and moments etc., should be able to read and understand this book. However, much of its contents may seem abstract unless the reader also has some previous experience with strength of materials and basic structural mechanics.

What this Book Is About—and what it Is Not

Part I, Continuum Mechanics

This book is about continuum mechanics and its central position in all our theories and methods for analysis of solids and structures. Although Part I is called Continuum Mechanics this book is not intended as a textbook covering all available theories of continuum mechanics, just the most essential topics of continuum mechanics—at least the ones that I have found the most important.

For some reason, it seems to be the case that often the Principle of Virtual Work does not get the attention it deserves, both in its own right and as a solid foundation for theories concerned with Specialized Continua—for my definition of this term, see page 27—such as beams, plates and shells.

What Are these Other Parts About?

The justification for the Parts II–VI is two-fold in that, on the one hand they provide theories and results that are useful by themselves, and on the other hand they serve as examples of application of the general principles and fundamental expressions of Continuum Mechanics, Part I. It is my intention

\begin{footnotes}
\item[1.16] Personally, I don’t like the way some books on mechanics are organized, namely beginning with a long part of math which one may not see the need for until much later. I prefer to get on with the “real” subject of mechanics and look up the math I don’t know by heart.
\item[1.17] Actually, you may never appreciate it.
\item[1.18] In more than one case, a footnote contains important information, either of a technical nature or expressing my personal view on a particular subject.
\end{footnotes}
that, together, Parts I–VI form an entirety in that the same notation is used and, even more important, because the same fundamental approach lies beneath every chapter.

Part II, Specialized Continua

Theories for some specialized continua,\footnote{I.19 This is probably not a common terminology, but I consider it clear and useful.} namely beams and plates, are developed in Part II, but it is not meant as a book about beams and plates and other specialized continua.\footnote{I.20 Other specialized continua, such as shells are not treated.} This part furnishes examples of theories for—very important—specialized continua developed from the general principles mentioned above.

Part III, Beams with Cross Sections and Plates with Thickness

In Part II beams are viewed as one-dimensional bodies and plates as two-dimensional bodies, both with some cross-sectional properties that are assumed known in some way. In Part III approximate continuum mechanics solutions are utilized to determine axial, shear and bending stiffness of important types of cross-sections. In addition to this, unconstrained torsion is treated and the torsional stiffness of a number of cross-sections is found. Finally, in order to show that finite elements need not be based on energy or virtual work a finite element procedure, which takes the value of the stress function for torsion as the unknowns, is developed.

Part IV, Buckling

The subject of buckling, including initial postbuckling and imperfection sensitivity, of linear elastic structures is covered to some extent in Part IV, but this book is not a book about stability and instability. Elastic-plastic buckling is studied via the so-called Shanley model column which provides a foundation for understanding this difficult subject. This part as a whole introduces fundamental concepts and, based on the general principles of continuum mechanics, it establishes some very important theories and methods relevant for the study of stability of structures.

Part V, Introduction to the Finite Element Method

In Part V, this useful tool is introduced, but this is not a finite element book. However, the method is introduced on the basis of the general principles of continuum mechanics. Some causes for errors in usual finite element analyses are investigated, and ways to deal with such problems are given.
Part VI, Mathematical Preliminaries

Part VI, contains some mathematical subjects, e.g. *Variational Principles*, and a number of fundamental relations and expressions of continuum mechanics formulated in terms of a compact notation, the so-called *Budiansky-Hutchinson Notation*, but this book is not about mathematics. This part merely serves to introduce the notation and some fundamental expressions that are employed in the rest of the book.

Some Comments on Notation

For any particular purpose, a certain notation may be superior to others, yet cumbersome for most other purposes. Here, we shall employ two notations which I have found to be convenient in the present connection. First, we encounter a notation that entails indices. This notation is very useful for deriving the equations of deformation and stress of a three-dimensional continuum or a plate, i.e. to establish equations for a three-dimensional continuum or a two-dimensional, specialized one. It is, however, rather awkward in connection with general statements, such as the *Principle of Virtual Work*, where we shall employ the so-called *Budiansky-Hutchinson Notation*. Both notations will be introduced in detail later, see Part VI.

Some Comments on Length

Some sections may seem too long because I cite formulas from other sections verbatim instead of just giving a reference—which, by the way, is done in most cases. This might seem in conflict with my wish to utilize compact notations, but the reason is that in some cases it proves to be very disturbing to have to flip from one page to another and then back to the original one in order to see the referenced formulas. Back when I was a graduate student I read Muskhelishvili’s book *Some Basic Problems of the Mathematical Theory of Elasticity* (Muskhelishvili 1963), which is a book of approximately 700 pages, in a fairly short time, and in that book it is rather the rule than the exception that earlier formulas are cited verbatim.

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1. We shall only use subscripts and avoid superscripts, except as labels, because the description will be in *Cartesian Coordinates* where the distinction between co- and contravariant components of tensors disappears.
2. Again, I have made choices that some readers may not like.
3. I am a slow reader, but it only took me about a month to read the first 550 pages.