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Carleman Estimates and Applications to Inverse Problems for Hyperbolic Systems
Preface

In this book, focusing on hyperbolic systems, we give self-contained descriptions of

- derivations of Carleman estimates;
- methods for application of Carleman estimates to stability of inverse problems.

Confining ourselves to equations of hyperbolic type, we survey previous and recent results concerning the applicability of Carleman estimates.

We do not intend to pursue any general treatment of the Carleman estimates themselves; rather by arguing in a direct manner, we mainly aim to demonstrate the applicability of Carleman estimates to inverse problems. In many places, we choose direct arguments based on basic calculus, rather than more general sophisticated methods. Because inverse problems are strongly connected with the respective partial differential equations under consideration and, for example, we have to specify unknown coefficients more concretely, and the direct method is more relevant for inverse problems. Moreover, we do not intend the current book to be encyclopedic in any sense, and the references are limited.

Some part is based on a one-semester course delivered at the Graduate School of Mathematical Sciences of The University of Tokyo by the first author when he was invited there as full professor in 2011–2012.

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Orientation

- Chapter 1 should be helpful for readers who are interested in quickly understanding the essence of Carleman estimates and applications to inverse problems.
- From Chap. 3 on, we give more general treatments. To this purpose, we need some material on Riemannian manifolds, presented in Chap. 2, where we give the minimal necessary account.
- Readers who prefer to become familiar faster with a direct but more comprehensive approach to Carleman estimates and applications can be advised to skip first Chaps. 2 and 3 and go straight to Chaps. 4 and 5. Then, whenever necessary, they can refer to corresponding parts in Chaps. 2 and 3.

Notations

\[ \delta_{ij} = \begin{cases} 1, & i = j; \\ 0, & i \neq j. \end{cases} \]

\(|x|\): the Euclidean norm of a vector \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \).

\((x_1, \ldots, x_n)^T\): the transpose of the vector \( x \in \mathbb{R}^n \).

\((x \cdot y)\): the scalar product of \( x, y \in \mathbb{R}^n \).

\((f, h) = \int_D f(x)h(x)dx\), where \( f, h \) are real-valued functions and \( D \) is a domain under consideration.

\( L^2(D) \): the space of real-valued functions \( f \) satisfying \( \int_D |f(x)|^2dx < \infty \).

\( \| \cdot \|_{L^2(D)} \): the norm in the space \( L^2(D) \). If there is no possibility of confusion, then we simply write \( \| \cdot \| \).

\( (\cdot, \cdot)_{L^2(D)} \): the scalar product in the space \( L^2(D) \). If there is no possibility of confusion, then we simply write \( (\cdot, \cdot) \).

\( M \): Compact smooth Riemannian manifold.

\( \partial M \): the boundary of \( M \), \( \Sigma_0 \): subboundary of \( M \).
\langle X, Y \rangle = \langle X, Y \rangle_g$: the scalar product for $X, Y \in T_x M$: the tangent space on Riemannian manifold $M$ with metric $g$.

$|X| = |X|_g := \sqrt{\langle X, X \rangle}$

$I_n$: the $n \times n$ identity matrix.

$\nu(x)$: the unit outward normal vector to the boundary under consideration.

$u' := \partial_i u = \frac{\partial u}{\partial x^i}, \partial_k u = \frac{\partial u}{\partial x^k}, \; k = 1, \ldots, n$. 

$\nabla_g, \nabla_g^2, \text{div}_g, \Delta_g$: the gradient, the Hessian, the divergence, the Laplace–Beltrami operator with the metric $g$ respectively.

$\nabla := \nabla_{I_n}, \nabla^2 := \nabla_{I_n}^2, \text{div} := \text{div}_{I_n}, \Delta := \Delta_{I_n}$: the gradient, the Hessian, the divergence, the Laplace–Beltrami operator with the Euclidean metric $I_n$ respectively.

$\partial_N u = (\nabla_g u \cdot \nu)$.

$
\begin{cases}
\text{In Chap. 1} \\
Q = (0, \ell) \times (0, T) \text{ or } Q = \Omega \times (0, T), 
Q_\pm = (0, \ell) \times (-T, T).
\end{cases}$

$\begin{cases}
\text{In Chaps. 2–10} \\
Q = M \times (0, T), 
\Sigma = \partial M \times (0, T), 
\Sigma_0 = \Gamma_0 \times (0, T), 
Q_\pm = M \times (-T, T), 
\Sigma_\pm = \partial M \times (-T, T), 
\Sigma_{0, \pm} = \Gamma_0 \times (-T, T).
\end{cases}$

$\begin{cases}
dx = (\det g)^{\frac{1}{2}} dx_1 \cdots dx_n, \quad \text{in a Riemannian manifold } M, \\
dx = dx_1 \cdots dx_n, \quad \text{in a bounded closed domain } M \subset \mathbb{R}^n
\end{cases}$