Part III

P/T-Net Synthesis
A uniform theory of Petri net synthesis, parametric on the type of nets, has been presented in Part II of this book. For each type of nets \( \tau = (S_\tau, E_\tau \Delta_\tau) \), the \( \tau \)-regions of an initialized transition system \( A = (S, E, \Delta, s_0) \) have been defined as transition system morphisms from \((S, E, \Delta)\) to \((S_\tau, E_\tau, \Delta_\tau)\). Such \( \tau \)-regions may equivalently be seen as \( \tau \)-nets with one place. The initialized transition systems or languages that may be realized exactly by \( \tau \)-nets have been characterized by separation axioms expressed in terms of \( \tau \)-regions. Moreover, it has been shown that initialized transition systems and languages may always be realized by optimal \( \tau \)-nets that over-approximate their behaviour. In any case, exact or approximate net realizations are synthesized from a set of \( \tau \)-regions by gluing the induced one-place \( \tau \)-nets to transitions.

More precisely, we recall that for any type of nets \( \tau \) and for any initialized transition system \( A \), the \( \tau \)-regions \( r \in R_\tau(A) \) are in bijective correspondence with the one-place \( \tau \)-net systems \( N_r \) such that \( A \leq RG_\tau(N_r) \) (or equivalently, \( A \cong A \times RG_\tau(N_r) \)). The correspondence maps any region \( r \) to the net system with the place \( r \), the flow relation \( F(r, e) = r(e) \) for every event \( e \), and the initial marking \( M_0(r) = r(s_0) \) where \( s_0 \) is the initial state of \( A \). Further, \( A \) may be realized by a \( \tau \)-net system if and only if \( A \cong RG_\tau(SN_R(A)) \) for some admissible set of \( \tau \)-regions \( R \subseteq R_\tau(A) \), where \( SN_R(A) \) is the \( \tau \)-net system formed by gluing to common transitions all net systems \( N_r \) with \( r \in R \). Finally, the least over-approximation of \( A \) by the reachability graph of a \( \tau \)-net system is \( RG_\tau(SN(A)) \), where \( SN(A) \) is the net synthesized from all \( \tau \)-regions of \( A \). The above facts apply in particular to (pure or impure) P/T-nets and P/T-regions. In the sequel, we rely on specialized versions of Theorems 5.8, 5.3, 5.12 established in Part II of this book, as follows.

**Theorem I.** Let \( A = (S, E, \Delta, s_0) \) be an initialized transition system. Then \( A \cong RG(N) \) for some (pure) P/T-net \( N \) iff \( A \cong RG(SN(A)) \), where \( SN(A) \) is the net synthesized from all (pure) P/T-regions of \( A \), iff all separation problems \( \{s, s'\} \) or \( \{s, e\} \) in \( A \) can be solved by (pure) P/T-regions of \( A \). In this case, \( A \cong RG(SN_R(A)) \) for any set of (pure) P/T-regions \( R \subseteq R(A) \) such that SSP(R) and ESSP(R) hold in \( A \).

**Theorem II.** Let \( A = (S, E, \Delta, s_0) \) be an initialized transition system and \( N \) be a (pure) P/T-net with the set of transitions \( E \). Then \( A \leq RG(N) \) iff \( N \leq SN(A) \), where \( SN(A) \) is the net synthesized from all (pure) P/T-regions of \( A \).

**Theorem III.** Let \( L \subseteq E^* \) be a non-empty prefix-closed language. Then \( L = L(N) \) for some (pure) P/T-net \( N \) with the set of transitions \( E \) iff \( L = L(SN(L)) \), where \( SN(L) \) is the net system synthesized from all (pure) P/T-regions of \( L \), iff all separation problems \( \{w, e\} \) with \( w \in L \) and \( w \notin L \) can be solved by (pure) P/T-regions of \( L \). In this case, \( L = L(SN_R(L)) \) for any set of (pure) P/T-regions \( R \subseteq R(L) \) such that ESSP(R) holds in \( L \).

For finite types of nets \( \tau \), effective procedures for the decision of the exact net realization problem, and for the synthesis of optimal net realizations,
follow directly from the uniform theory. However, the theory is not effective for infinite types of nets, and in particular for the type $\tau_{PT}$ of the P/T-nets. Indeed, $\tau_{PT}$ has the set of states $S = \mathbb{N}$, reflecting that a place may be marked with an arbitrary non-negative integer, counting the tokens in that place. Theorems I and III do not lead directly to decision and synthesis algorithms solving the P/T-net realization problem for initialized transition systems or for languages, seen as transition systems in which state separation is not required. Theorem II does not lead either to synthesis algorithms computing the least P/T-net over-approximation of a transition system or of a language. The difficulty lies in the fact that (pure or impure) P/T-regions form infinite sets even for a finite initialized transition system. This difficulty will be overcome in the forthcoming chapters by exploiting the linear-algebraic properties of P/T-regions.