Part III
RELATIVE COMPUTABILITY
In the previous part we described how the world of incomputable problems was discovered. This resulted in an awareness that there exist computational problems that are unsolvable by any reasonable means of computing, e.g., the Turing machine. In Part III, we will focus on decision problems only. We will raise the questions, “What if an unsolvable decision problem had been somehow made solvable? Would this have turned all the other unsolvable decision problems into solvable problems?” We suspect that this might be possible if all the unsolvable decision problems were somehow reducible one to another. However, it will turn out that this is not so; some of them would indeed become solvable, but there would still remain others that are unsolvable. We might speculate even further and suppose that one of the remaining unsolvable decision problems was somehow made solvable. As before, this would turn many unsolvable decision problems into solvable; yet, again, there would remain unsolvable decision problems. We could continue in this way, but we would never exhaust the class of unsolvable problems.

Questions of the kind “Had the problem $Q$ been solvable, would this have made the problem $P$ solvable too?” are characteristic of the relativized computability, a large part of the Computability Theory. This theory analyzes the solvability of problems relative to (or in view of) the solvability of other problems. Although such questions seem to be overly speculative and the answers to be of questionable practical value, they nevertheless reveal a surprising fact: Unsolvable decision problems can differ in the degree of their unsolvability. We will show that the class of all decision problems partitions into infinitely many subclasses, called degrees of unsolvability, each of which consists of all equally difficult decision problems. It will turn out that, after defining an appropriate relation on the class of all degrees, the degrees of unsolvability are intricately connected into a lattice-like structure. In addition to having many interesting properties per se, the structure will reveal many surprising facts about the unsolvability of decision problems.

We will show that there are several approaches to partitioning the class of decision problems into degrees of unsolvability. Each of them will establish a particular hierarchy of degrees of unsolvability, such as the jump and arithmetical hierarchies, and thus offer yet another view of the solvability of computational problems.