Part IV
Optimal Portfolios and Extreme Risks

Portfolio diversification is a basic approach to the reduction of non-systemic risks. The mean-variance theory of Markowitz (1952) was introduced in order to explain the effects of portfolio diversification concerning reduction of risk given the reward or in order to maximize the reward given the risk (see also Markowitz (1991)). The reward of a portfolio $\xi \cdot X$, where $\xi$ is a portfolio vector and $X$ is the vector of risks in the portfolio is measured by the mean $E\xi \cdot X$ while the risk is measured in this theory by the variance $\text{Var}(\xi \cdot X) = \xi^\top \Sigma \xi$, where $\Sigma = \text{Cov}(X)$ is the covariance matrix of $X$.

The portfolio optimization problem is then the problem to maximize the reward given bounds on the risk over all admissible portfolio vectors $\xi$

$$E\xi^* \cdot X = \sup_{\xi} \{ E\xi \cdot X ; \text{Var}(\xi \cdot X) \leq v_0 \}$$

or the related problem of minimizing the risk given the reward, i.e.

$$\text{Var}(\xi^* \cdot X) = \inf_{\xi} \{ \text{Var}(\xi \cdot X) ; E\xi \cdot X \geq r_0 \}$$

or to optimize a related risk reward functional like

$$\frac{E\xi \cdot X}{(\text{Var}(\xi \cdot X))^{1/2}} = \sup_{\xi} .$$

As in the chapter on optimal risk allocation it is also for the portfolio diversification problem well motivated to consider more relevant risk measures $\Psi$ replacing the variance and on the other hand also more stable versions of measuring the reward $\xi \cdot X$ of a portfolio.

In this chapter we consider an approach to this diversification problem for portfolios with heavy-tailed components. Heavy-tailed portfolios even with infinite mean are common in several branches of applications in insurance or financial risks.
(see e.g. Moscadelli (2004) or Nešlehová et al. (2006b) for empirical evidence). So mean-variance measures or convex risk measures will not be applicable. In the framework of extreme value theory, in particular the theory of multivariate regular variation, we consider the portfolio diversification problem. Portfolio losses are compared by their sensitivity w.r.t. extremal risk events. The aim is to determine portfolios in an optimal way such that they avoid extremal risk events as much as possible.

To this aim a functional \( \gamma_{\xi} = \gamma_{\xi}(\alpha, \Psi) \) is introduced which depends on the vector of portfolio weights \( \xi \) and on the distributional parameters \( \alpha, \Psi \) where \( \alpha \) is the tail index and \( \Psi \) is the spectral measure arising from the multivariate regular variation assumption. It is argued that the “extremal risk index” \( \gamma_{\xi} \) describes the sensitivity of the portfolio \( \xi \cdot X \) concerning extremal risk events. The optimal diversification problem thereby is reduced to the optimization of the extremal risk index \( \gamma_{\xi} \) w.r.t. \( \xi \).

An interesting effect is obtained by observing that for models with \( \alpha < 1 \), i.e. models with infinite mean, diversification does not improve the portfolio but it makes the portfolio worse. For \( \alpha > 1 \) we obtain the expected positive diversification effects while the case \( \alpha = 1 \) is indifferent. We also introduce empirical versions (estimators) of the optimal portfolio and the extremal risk index and establish consistency and asymptotic normality.

The second part of this chapter is concerned with a comparison of different stochastic models w.r.t. the asymptotic portfolio losses. The corresponding notion of “asymptotic portfolio loss order” is introduced and several sufficient conditions are given in order to verify this order in various classes of examples. Also connections to several further stochastic orders are elaborated.

For \( \alpha < 1 \) stronger positive dependence typically decreases extremal risk while for \( \alpha \geq 1 \) stronger positive dependence increases risk. This phenomenon for \( \alpha \geq 1 \) is concordant with the behaviour of convex risk measures and the related convex order \( \preceq_{\text{cx}} \) (see Section 3.1) for integrable risks. The examples include elliptical distributions and multivariate regularly varying models with Gumbel, Archimedean, and Galambos copulas.