The optimal risk allocation respectively risk sharing problem has a long history in mathematical economics and insurance and is of considerable practical and theoretical interest. The basic problem can be described as follows. We consider a “market” given by a probability space $(\Omega, \mathcal{A}, P)$, $n$ economic agents (traders) supplied with risks $X_1, \ldots, X_n$ and possibly different risk measures $\varrho_1, \ldots, \varrho_n$ for the evaluation of their risks. The problem is to redistribute the total risk $X = \sum_{i=1}^{n} X_i$ to the traders by a reallocation $\sum_{i=1}^{n} Y_i$ such that the risk vector $(\varrho_i(Y_i))$ is Pareto optimal in the class of all admissible allocations of $X$ or such that the total risk $\sum_{i=1}^{n} \varrho_i(Y_i)$ is minimal under all admissible allocations. In some variants of the problem additional well-motivated constraints are put on the allocation problem as for example side constraints of the form $Y_i \geq A_i$ or $Y_i \geq X_i - A_i$, which limit the magnitude of exchange of agent $i$ or upper bounds $Y_i \leq X_i + B_i$ respectively $\leq A_i + B_i$ which protect the liquidity of the individual traders. An alternative restriction is the “individual rationality condition”

$$\varrho_i(Y_i) \leq \varrho_i(X_i) \quad \text{or} \quad Y_i \geq_{icx} X_i;$$

only those changes are admissible and acceptable for trader $i$ which are preferable compared with the actual risk $X_i$.

The classical examples of risk exchange are from insurance where a risk $X$ is redistributed between two traders, an insurer and a reinsurer, both supplied with their own risk measures or equivalently with their specific utility functions. The classical insurance respectively reinsurance contracts like linear quota sharing or stop-loss contracts can be derived in this way as optimal reallocation (reinsurance) contracts.

In a series of classical papers Borch (1960a,b, 1962), Du Mouchel (1968), and Gerber (1978) showed that based on utility functions Pareto optimal risk exchanges can be characterized and in many cases lead to familiar linear quota-sharing of the total pooled losses or to stop-loss contracts and to mixtures of both. Solutions are however typically not uniquely determined which may lead to the necessity to arrange substantial side payments in order to make these solutions acceptable.
In several papers authors have added game-theoretic considerations or additional concepts (like the concept of fairness) to arrive at a specific element in the set of Pareto optimal rules (see Borch (1960b), Lemaire (1977), and Bühlmann and Jewell (1979)).

Since risk pools redistribute only actual losses and possibly the associated premiums but not the individual wealth of the company it is natural as mentioned above to include side constraints in the exchange protocol. The importance of side constraints has been suggested by Borch (1968) and has formally been introduced and applied in Gerber (1978, 1979).

Several authors have extended the framework to include the presence of background risk and have considered the allocation problem also in the context of financial risks (see Leland (1980), Chevallier and Müller (1994), and Barrieu and El Karoui (2004, 2005), Dana and Scarsini (2007), Chateauneuf et al. (2000), Denault (2001) and references therein). A main motivation comes from portfolio optimization problems which can be considered in a joint market model as generalized form of the risk exchange problem. Here more general exchange mechanisms described by trading strategies are considered. Also more general types of risk measures (distortion type, coherent, convex, comonotone risk measures) have been considered for the allocation problem. For the background literature on risk measures and their applications to finance and insurance we refer to Part II as well as to Deprez and Gerber (1985), Kaas et al. (2001), Delbaen (2000, 2002), and Föllmer and Schied (2011).

In Chapter 10 we consider the unrestricted optimal risk allocation problem. The problem is to characterize optimal allocations of a risk $X \in L^p(P)$ to the $n$ traders, i.e. to determine solutions of the problem

$$\sum_{i=1}^n q_i(X_i) = \inf$$

(III.1)

under all allocations of $X$ to the traders, i.e. under all decompositions $X = \sum_{i=1}^n X_i$, $X_i \in L^p(P)$. Solutions of the risk allocation problem are not unique but in fact are given under an equilibrium condition by the set of all Pareto optimal allocations, as follows from a general separation argument and the translation invariance of the $q_i$ (see Gerber (1979, pp. 88–96)). Thus the optimal allocation problem can be interpreted as a problem to minimize the total risk of a risk sharing contract but also as a basic tool to determine Pareto optimal allocations. The value of the optimal allocation problem is given by the infimal convolution $\hat{\varrho} = q_1 \wedge \ldots \wedge q_n$ defined for $X \in L^p(P)$ by

$$\hat{\varrho}(X) = \inf \left\{ \sum_{i=1}^n q_i(X_i); \quad X_i \in L^p(P), \sum_{i=1}^n X_i = X \right\}.$$  

(III.2)
We show that the general formulation of the optimal risk allocation problem in (III.1) and (III.2) makes sense only under a Pareto equilibrium condition (E). In vague form (E) can be stated as follows:

A market is in equilibrium if in a balance of supply and demand it is not possible to lower the risk of some traders without increasing that of some other traders. This equilibrium condition (E) has been characterized for coherent risk measures $\varrho_1, \ldots, \varrho_n$ in Heath and Ku (2004) and in Burgert and Rü (2005/2008) in terms of the scenario measures of the $\varrho_i$. An extension of this characterization is given in Burgert and Rü (2006b). The Pareto equilibrium condition (E) implies that $\varrho$ is a convex risk measure. In the coherent case also the converse relation holds true.

We discuss various monotonicity results for optimal risk allocations, stating that optimal allocations $X = \sum_{i=1}^n X_i$ of $X$ can be found in the class of allocations $X_i$ such that $X_i$ are comonotone to $X$ respectively where $X_i = f_i(X)$ are monotonically increasing functions of $X$. There are several related monotonicity results on the construction and design of optimal options in mathematical finance which we discuss briefly.

We introduce a class of well-motivated restrictions on the allocations which leads to a meaningful version of the allocation problem also without an equilibrium condition. The idea of introducing this class of restrictions is connected with a similar idea in portfolio theory, where one considers (lower bounded) admissible strategies in order to exclude strategies which allow arbitrage. As a consequence we obtain a convex risk measure – the convex infimal admissible convolution risk measure – which describes the optimal total admissible risk $\sum_{i=1}^n \varrho_i (X_i)$ avoiding the possibility of risk arbitrage.

Chapter 11 is concerned with some generalizations of the classical characterization results of optimal allocations due to Borch (1960b) and others. These allow to derive several of the well-known reinsurance contracts as optimal solutions of the allocation problems.

In the final part of this chapter we give extensions to the risk allocation problem for portfolio vectors $X \in \mathbb{R}^d$. Optimal allocations in this context are connected with the worst case dependence structure and $\mu$-comonotonicity. These connections are inherited from the representation results for risk measures for portfolio vectors.

Chapter 12 is concerned with applications of the dependence bounds in Part I to the construction of optimal contingent claims and portfolios and also of optimal (re-)insurance contracts. We deal in particular with the construction of efficient portfolios and of optimal robust insurance contracts.