Part II

Risk Measures and Worst Case Portfolios

The purpose of a risk measure is to specify the “riskiness” of an insurance contract $X$ or of a portfolio held by a financial institution, by an index $\phi(X)$. In insurance the risks are typically positive and we use the notation $\Psi(X)$ for risk measures which are monotone in the usual sense; $X \leq Y$ implies that $\Psi(X) \leq \Psi(Y)$. In finance, the losses are typically negative and $X \leq Y$, therefore, implies that the risk of $X$ is larger than the risk of $Y$, i.e. $\varrho(X) \geq \rho(Y)$ for the risk measure denoted by $\rho$ in this context.

Risk measures have a long tradition in insurance going back to the 1970s (see Bühlmann (1970)). They serve to calculate premiums of insurance contracts and the discussion of the “premium calculation principles” is a basic part of risk management in insurance. In mathematical finance risk measures were introduced in the late 1990s and their relevance for handling financial risk has become a predominant subject.

For an introduction to risk measures we refer in particular to Gerber (1979), Goovaerts et al. (1984), Kaas et al. (2001), and Denuit et al. (2005) for the actuarial side and for the financial risk theory to Föllmer and Schied (2011), Delbaen (2002), and Pflug and Römisch (2007). The main subject in this chapter is the investigation of risk measures for risk vectors and portfolios. Besides the risk coming from the components $X_i$ of the risk vector $X$ we are in particular interested in describing the risk coming from possible dependence between the components. Risk measures should evaluate this additional “dependence risk” in a correct way. This postulate is not easy to formulate and some early postulates in risk theory turned out in the more recent literature to need essential modification and specification.

One prominent example of this type of changing view is the postulate on risk measures that “diversification of risk should be encouraged by the risk measure $\rho$”, i.e. the postulate of “subadditivity”

$$\varrho \left( \sum_{i=1}^{n} X_i \right) \leq \sum_{i=1}^{n} \varrho(X_i) \quad (\text{II.1})$$
or the related weaker variant of “convexity”
\[ \varphi \left( \sum_{i=1}^{n} \alpha_i X_i \right) \leq \sum_{i=1}^{n} \alpha_i \varphi_i (X_i). \]  
(II.2)

This kind of “economic axiom” was used to argue against the classical value at risk \( \text{VaR}_\alpha \) risk measure which is neither subadditive nor convex and to propose to use instead the expected shortfall risk measure \( \text{ES}_\alpha (x) \) defined (in the insurance version) as excess of loss
\[ \text{ES}_\alpha (X) = E (X - \text{VaR}_\alpha (X))_+ \]  
(II.3)
or some variant of it. The expected shortfall measures the magnitude of loss above the “default level” \( \text{VaR}_\alpha (X) \).

Diversification however is in general not a useful and justified postulate. If we take for example a portfolio \( X = (X_1, \ldots, X_n) \) of heavy tailed components \( X_i \) say \( X_i \) are symmetric \( \alpha \)-stable, i.e. the characteristic function of \( X_i \) is given by
\[ \varphi_{X_i}(t) = e^{-c|t|^\alpha} \]
with some constant \( c > 0 \). If we assume that \( (X_i) \) are independent, then the joint portfolio \( \sum_{i=1}^{n} X_i \) is distributed as \( n^{1/\alpha} X_1 \) or equivalently for the diversified portfolio \( \frac{1}{n} \sum_{i=1}^{n} X_i \) it holds that
\[ \frac{1}{n} \sum_{i=1}^{n} X_i \overset{d}{=} n^{1/\alpha - 1} X_1. \]  
(II.4)

For the proof of (II.4) note that
\[ \varphi_{\sum_{i=1}^{n} X_i}(t) = (\varphi_{X_1}(t))^n = e^{-cn|t|^\alpha} = \varphi_{n^{1/\alpha} X_1}(t). \]

In consequence in the practically relevant case \( \alpha < 1 \) the diversified portfolio is by a magnitude \( n^{1/\alpha - 1} \) worse than the undiversified. Any risk measure that would be convex on this model would evaluate the risk in a wrong way. For \( \alpha > 1 \) we have existence of first moments and a positive diversification effect arises in (II.4). Useful risk measures should reflect this effect. For bounded risks this effect does not appear but for non-integrable risks the axioms of convexity or subadditivity do not make sense in general. One principal problem of risk measures is that the risk distribution (respectively loss distribution) which gives the complete picture of the risk is difficult (or even impossible) to represent by an index \( \varphi(X) \) or \( \Psi(X) \).

To demonstrate the importance of describing the risk coming from dependence in the portfolio we consider the following simple example.
Example II.1 (Dependence effect in mixture models). Let \( X_i = \Theta Y_i + (1 - \Theta) Z_i \), \( 1 \leq i \leq 10^5 \) be a portfolio of \( 10^5 \) contracts, where \((Y_i), (Z_i)\) are iid, independent of \( \Theta \) and
\[
Y_i \sim \mathcal{B} \left( 1, \frac{1}{100} \right), \quad Z_i \sim \mathcal{B} \left( 1, \frac{1}{10000} \right), \quad \Theta \sim \mathcal{B} \left( 1, \frac{1}{100} \right)
\]
are binomial distributed. \( \Theta \) introduces a ‘small’ positive dependence in terms of correlation between the components of the \( X_i \). What is the influence of this small positive dependence on the risk of the joint portfolio \( S_n = \sum_{i=1}^n X_i \)?

Note that \( X_i \sim \mathcal{B} \left( 1, \frac{1}{10000} \right) \). Let \((W_i), 1 \leq i \leq 10^5\) be a related iid portfolio \( W_i \sim \mathcal{B} \left( 1, \frac{1}{10000} \right) \) and denote \( T_n = \sum_{i=1}^n W_i \) the risk of the joint iid portfolio. Then from the central limit theorem we get an approximation by the normal distribution
\[
T_n \sim N(100, 100).
\]

The distribution of the related mixture portfolio \( S_n \) is approximatively given by a mixture distribution
\[
S_n \sim \frac{1}{100} N(1000, 1000) + \frac{99}{100} N(100, 100).
\]

In consequence the iid portfolio \((W_i)\) is safe w.r.t. the excess of loss with retention limit \( t = 150 \) (which is the 5\( \sigma \) domain) \( E(T_n - 150)_+ \) is in the range of \( 10^{-8} \). But \( E(S_n - 150)_+ \approx 8.5 \) is of considerable magnitude. Note that \( P(S_n > 150) \sim \frac{1}{100} \).

Thus there exists a considerable risk probability for the mixture model, which is induced by the small positive dependence in the sample. \( \diamond \)

After a short introduction to examples and properties of risk measures of real risks we consider their robust representation and continuity properties on \( L^p \)-spaces in Chapter 7. Right from the beginning we allow for unbounded risks, since the typical models of risks are not bounded. In Chapter 8 we concentrate on the structure, examples, and properties of risk measures for portfolio vectors. It turns out that their representation is closely connected with mass transportation problems. This connection was first described in Rü (2006). In particular one finds that the role of the expected shortfall and more generally of the spectral risk measures is taken in the multivariate case by the “max correlation risk measures” which are defined by a mass transportation problem.

In contrast to the real case there is no longer a useful notion of comonotonicity which describes a “universal” worst case dependence structure in the case of risk vectors. Instead we obtain depending on a risk measure \( \varrho \) (or \( \Psi \)) a worst case dependence structure, which one could call in analogy to the one-dimensional case “\( \varrho \)-comonotone dependence structure”. A general characterization of this worst case dependence structure was given in Rü (2012a). For the special case of max-correlation risk measures see also Ekeland et al. (2010). We also describe the worst case diversification effect measured by
\[
\inf \left\{ \sum_{i=1}^{n} q(X_i) - q \left( \sum_{i=1}^{n} X_i \right) \right\} \tag{II.5}
\]

respectively

\[
\inf \left\{ \frac{1}{n} \sum_{i=1}^{n} q(X_i) - q \left( \frac{1}{n} \sum_{i=1}^{n} X_i \right) \right\}, \tag{II.6}
\]

the inf being over all possible dependence structures. The max correlation risk
measures are (up to some regularity condition) the only risk measures where the
worst case diversification effect is zero.