In financial and insurance risk management as in many further areas of probabilistic modelling as for example in network analysis there are typically several sources of risks. In insurance there are the different contracts held by an insurance company, in finance the portfolio held by a bank is composed of many risky assets. The various risk components are typically not independent of each other. In consequence it became necessary to develop the proper tools to describe the relevant statistical models for dependent variables and to analyse their properties.

There are two basic tasks to face this situation. The first one is the problem of “dependence modelling”. Here we are given a vector of risks \( X = (X_1, \ldots, X_n) \) where the components \( X_i \) have known distributions \( P_i \). \( X_i \) themselves might be one-dimensional or higher dimensional risks. The joint distribution \( P^X \) of \( X \) then is an element of the “Fréchet class” \( \mathcal{M}(P_1, \ldots, P_n) \) of all probability measures \( P \) on the product space which have marginals \( P_i \), i.e. \( P^{\pi_i} = P_i \), \( 1 \leq i \leq n \), where \( \pi_i \) are the projections on the \( i \)-th component. To describe models for the dependence structure of \( X \) is equivalent to describe (parametric) submodels of the Fréchet class \( \mathcal{M}(P_1, \ldots, P_n) \). In the case of one-dimensional marginals with distribution functions \( F_i \) also the notion \( \mathcal{F}(F_1, \ldots, F_n) \) is used for the Fréchet class of corresponding distribution functions. The basic notion of copula introduced by Sklar aims to separate the description of the dependence part of a distribution and the marginal part in the case of one-dimensional marginals. In consequence dependence models are given by specifying the marginals and specifying the corresponding copula models.

The second main subject of describing “dependence” are notions of dependence orderings, corresponding dependence measures and the description of bounds on the possible influence of dependence on certain risk functionals like the risk of the joint portfolio in finance or in insurance. Basic results in this direction are the classical Hoeffding–Fréchet bounds which specify upper and lower bounds for the joint distribution function \( F = F^X \) of the risk vector \( X \). In this connection the notion of “comonotonicity” is used to describe the worst case dependence structure for one-dimensional marginal risks.
In Chapter 1 we give an introduction to the basic notion of copulas and some related useful tools for the construction of probability models. We introduce the distributional transform, the quantile transform, and their multivariate variants, the multivariate distributional transform and the multivariate quantile transform, which give a basic construction and simulation method and which extend the classical Rosenblatt transform. We discuss applications to copulas, to the conditional tail expectation, and to the empirical copula process, which is a basic statistical tool for dependence analysis.

In Chapter 2 we describe various problems of generalized Fréchet bounds. The basic tool to deal with these problems is a duality theorem describing the range of influence of dependence on an integral functional by a dual representation. In case $n = 2$ this dual representation is closely related to a corresponding result in mass transportation theory going back in its earliest version to Kantorovich. Based on this duality result sharp bounds on the influence of dependence on various risk functionals can be given. Also the description of optimal couplings is closely related to this duality result. We describe in detail in Chapters 3 and 4 resulting bounds for the value at risk (equivalently for the distribution function) and for the excess of loss of the joint portfolio. The result for the excess of loss explains the universal worst case character of the comonotonic dependence structure. Under restrictions on the class of possible dependence structures one can give strongly improved bounds for the risk functionals. We discuss the restriction to positive dependent risk vectors and the restriction on the dependence structure induced by higher order marginals in Chapter 5. Finally in Chapter 6 the risk bounds can in some cases be made more informative by the method of dependence orderings which allow to describe some structure of the influence of dependence within the Fréchet classes and to compare different models concerning the risk induced by their internal dependence properties.