Microstructured Materials: Inverse Problems
Preface

This book is about the mathematical treatment of inverse problems related to material characterisation. We realised that we have to write a much longer treatise than usual research papers in order to describe our ideas in a proper informative way.

There is always a question about theory and practice like the iconic “the chicken and the egg” causality dilemma. We might say that an apple fell first and then Isaac Newton presented a theory. On the other hand, the ideas of Paul Dirac are excellent examples of how theory precedes practice. And certainly much has been written about the balance of theory and practice. We hope that, although we are theorists ourselves, we have taken the idea of balance seriously.

Indeed, this way or another, in material science the will and the need to look into materials for determining the physical or geometrical properties or residual stresses goes back to the history. The audible ring of a Damascus sword blade or a church bell was an indication of quality, for example. And farmers could estimate the ripeness of a water-melon by tapping it and listening to the sound. However, contemporary technology and materials need more “advanced” techniques, and the information we are interested in is more sophisticated.

Here we deal with Non-Destructive Evaluation (NDE) of material properties with the focus on microstructured materials. The tool for this is a wave which propagates through a specimen or a structural element. A wave travelling in materials and propagating over a certain distance collects and “encodes” the information on its path. The problem is how to “decode” this information.

For many practical purposes the material is assumed to be homogeneous. In this case the “decoding” is rather simple—the flight time of a wave (a signal) permits the sound velocity to be determined and, from that, some information about the material properties (density, modulus of elasticity) can be deduced. However, the contemporary materials are much more complicated and their internal structure, i.e., the microstructure, affects the result. Moreover, there is a need to evaluate the properties of the microstructure. The list of such microstructured materials widely used in modern technology is long: alloys, ceramics and composites, functionally graded materials, granular materials and nano-materials, biomedical and optical materials, etc. Consequently, the methods of the NDE must be based on the adequate analysis of the
effects which will be “encoded” by a propagating wave, and the following “decoding” must be properly built up. Following these ideas, the mathematical modelling of waves in microstructured solids must give a well-grounded basis for the analysis. The focus is on dispersion which is the leading effect for waves in such materials. This is exactly the starting point for the book. After proposing (with suitable assumptions) a sound mathematical model, we discuss the number of unknowns to be inversely identified, then establish the uniqueness of a solution and only then propose the ideas for solving the inverse problems. We hope that such a consistent approach will build a proper basis for practical applications.

We have published several research papers on this topic over the last six years (see references). However, the book is not simply a collection of these papers but much is rewritten to cast the material into a unified whole and much is added in order to cement the ideas.

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