# Table of Contents

**J. Richard Büchi:**

**The Monadic Second Order Theory of $\omega_1$**  
1

## 1. Monadic second order theories; the prenex form lemma  
4  
The language, standard models, monadic second order theories  
4  
Prenex forms, sup conditions  
9  
Standard axiom systems, relativization  
17

## 2. A method for eliminating set-quantifiers  
21  
Decision method for monadic second order theories, transition system, finite automata  
21  
Decidability results for monadic second order theories  
27  
Open problems  
37

## 3. The working of the method in the case of $\omega_0$  
40  
Item 1: Transition systems, runs  
40  
Item 2: The complementation lemma, the subset construction in the finite case, merging, automata normal form  
42  
Item 3: The decision-method  
54  
Item 2': The subset-construction at $\omega$, deterministic automata normal form  
56

## 4. The subset-construction extended up to $\omega_1$  
71  
Deterministic automata normal form for countable ordinals  
71
The input-free case: $\omega_1$-spectrum, associated operators, expansion, head, tail, character, absorption automata, the Tarski-Lindenbaum algebra, equivalent countable ordinals

Decision method for $\text{MT}[\text{co}]$ and $\text{MT}[\alpha], \alpha < \omega_1$

5. **The filters of closed cofinal sets**

The filter $\mathcal{J}_1^\infty$ of closed cofinal sets, the dual ideal $\mathcal{O}_1^\infty$, derivatives, the weakness of sup-transition-systems

$P_{\omega_1}/\mathcal{J}_1$ is atomless, $\sup_{\mathcal{J}_1}$-conditions

6. **A decision method for $\text{MT}[\omega_1, <]$**

Complementation, automata normal form, decision method

Bibliography
WLO, relativizations of formulas and structures, limits, cofinality, ACC

The standard axiom system $\phi_0$

Relativization of $\phi_0$

$S_k, T_k, q, \sum_{n_k}, ..., n_1, \sum_\alpha$, the standard axiom systems

$A_\alpha$, definability of ordinals

SPLICE, the axiom systems $\phi_0$ and $A_\alpha$

3. The automata normal form

(Formalization of section [Bü] 2.)

Derivations from WLO: Pigeon hole principle, set recursion, automata normal form plus prefix

4. The completeness of $A_\omega$

(Formalization of section [Bü] 3, item 2.)

Derivations from $A_\omega$: SPLICE, the axiom of definable choice, automata normal form, completeness

5. The deterministic automata normal form

(Formalization of section [Bü] 4, first half.)

Derivations from $\phi_0$: Deterministic automata normal form, terminal condition

6. $\phi_0$ and $A_\alpha, \alpha < \omega_1$, are axiom systems

(Formalization of section [Bü] 4, second half.)

Derivations from $\phi_0$: Properties of the operators $Z_0, F_1, V_1, S_1$. 
VI

$\Omega_\alpha$, red $\phi$, the completeness of $\overline{\alpha}$ and of $\overline{\varnothing}$. 188

7. The axiom system $\overline{\epsilon}_1$ for $MT[\omega_1]$ 195

(Formalization of sections [Bü] 5+6.)

The standard axiom system $\epsilon_1$. 195

Cofinal-closed and stationary sets, the filter $\mathcal{J}_1$, 195

atoms, the axiom system $\overline{\epsilon}_1$.

Derivations from $\overline{\epsilon}_1$: Automata normal form, completeness 196

8. The independence of the splicing axiom 205

Fundamental sequences and arrangements, $CS(\alpha)$ and $CA(\alpha)$, Church's results 205

CD, Litman's result 206

Levi's result, SPLICE independent from $\epsilon_0$ and $\overline{\alpha}$ 210

Hajek's result, SPLICE independent from $\epsilon_1$ 211

Problems 211

References 215