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Operator Functions and Localization of Spectra

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Preface

1. A lot of books and papers are concerned with the spectrum of linear operators but deal mainly with the asymptotic distributions of the eigenvalues. However, in many applications, for example, in numerical mathematics and stability analysis, bounds for eigenvalues are very important, but they are investigated considerably less than asymptotic distributions. The present book is devoted to the spectrum localization of linear operators in a Hilbert space. Our main tool is the estimates for norms of operator-valued functions. One of the first estimates for the norm of a regular matrix-valued function was established by I. M. Gel’fand and G. E. Shilov in connection with their investigations of partial differential equations, but this estimate is not sharp; it is not attained for any matrix. The problem of obtaining a precise estimate for the norm of a matrix-valued function has been repeatedly discussed in the literature. In the late 1970s, I obtained a precise estimate for a regular matrix-valued function. It is attained in the case of normal matrices. Later, this estimate was extended to various classes of nonselfadjoint operators, such as Hilbert-Schmidt operators, quasi-Hermitian operators (i.e., linear operators with completely continuous imaginary components), quasiunitary operators (i.e., operators represented as a sum of a unitary operator and a compact one), etc. Note that singular integral operators and integro-differential ones are examples of quasi-Hermitian operators.

On the other hand, Carleman, in the 1930s, obtained an estimate for the norm of the resolvent of finite dimensional operators and of operators belonging to the Neumann-Schatten ideal. In the early 1980s sharp estimates for norms of the resolvent of nonselfadjoint operators of various types were established, that supplement and extend Carleman’s estimates. In this book, we present the mentioned estimates and, as it was pointed out, systematically apply them to spectral problems.

2. The book consists of 19 chapters. In Chapter 1, we present some well-known results for use in the next chapters.

Chapters 2-5 of the book are devoted to finite dimensional operators and functions of such operators.

In Chapter 2 we derive estimates for the norms of operator-valued functions in a Euclidean space. In addition, we prove relations for eigenvalues of finite matrices, which improve Schur’s and Brown’s inequalities.
Although excellent computer softwares are now available for eigenvalue computation, new results on invertibility and spectrum inclusion regions for finite matrices are still important, since computers are not very useful, in particular, for analysis of matrices dependent on parameters. But such matrices play an essential role in various applications, for example, in the stability and boundedness of coupled systems of partial differential equations. In addition, the bounds for eigenvalues of finite matrices allow us to derive the bounds for spectra of infinite matrices. Because of this, the problem of finding invertibility conditions and spectrum inclusion regions for finite matrices continues to attract the attention of many specialists. Chapter 3 deals with various invertibility conditions. In particular, we improve the classical Levy-Desplanques theorem and other well-known invertibility results for matrices that are close to triangular ones. Chapter 4 is concerned with perturbations of finite matrices and bounds for their eigenvalues. In particular, we derive upper and lower estimates for the spectral radius. Under some restrictions, these estimates improve the Frobenius inequalities. Moreover, we present new conditions for the stability of matrices, which supplement the Rohrbach theorem.

Chapter 5 is devoted to block matrices. In this chapter, we derive the invertibility conditions, which supplement the generalized Hadamard criterion and some other well-known results for block matrices.

Chapters 6-9 form the crux of the book. Chapter 6 contains the estimates for the norms of the resolvents and analytic functions of compact operators in a Hilbert space. In particular, we consider Hilbert-Schmidt operators and operators belonging to the von Neumann-Schatten ideals.

Chapter 7 is concerned with the estimates for the norms of resolvents and analytic functions of non-compact operators in a Hilbert space. In particular, we consider so-called $P$-triangular operators. Roughly speaking, a $P$-triangular operator is a sum of a normal operator and a compact quasinilpotent one, having a sufficiently rich set of invariant subspaces. Operators having compact Hermitian components are examples of $P$-triangular operators.

In Chapters 8 and 9 we derive the bounds for the spectra of quasi-Hermitian operators.

In Chapter 10 we introduce the notion of the multiplicative operator integral. By virtue of the multiplicative operator integral, we derive spectral representations for the resolvents of various linear operators. That representation is a generalization of the classical spectral representation for resolvents of normal operators. In the corresponding cases the multiplicative integral is an operator product.

Chapters 11 and 12 are devoted to perturbations of the operators of the form $A = D + W$, where $D$ is a normal boundedly invertible operator and $D^{-1}W$ is compact. In particular, estimates for the resolvents and bounds for the spectra are established.
Chapters 13 and 14 are concerned with applications of the main results from Chapters 7-12 to integral, integro-differential and differential operators, as well as to infinite matrices. In particular, we suggest new estimates for the spectral radius of integral operators and infinite matrices. Under some restrictions, they improve the classical results.

Chapter 15 deals with operator matrices. The spectrum of operator matrices and related problems have been investigated in many works. Mainly, Gershgorin-type bounds for spectra of operator matrices with bounded operator entries are derived. But Gershgorin-type bounds give good results in the cases when the diagonal operators are dominant. In Chapter 15, under some restrictions, we improve these bounds for operator matrices. Moreover, we consider matrices with unbounded operator entries. The results of Chapter 15 allow us to derive bounds for the spectra of matrix differential operators.

Chapters 16-18 are devoted to Hille-Tamarkin integral operators and matrices, as well as integral operators with bounded kernels.

Chapter 19 is devoted to applications of our abstract results to the theory of finite order entire functions. In that chapter we consider the following problem: if the Taylor coefficients of two entire functions are close, how close are their zeros? In addition, we establish bounds for sums of the absolute values of the zeros in the terms of the coefficients of its Taylor series. They supplement the Hadamard theorem.

3. This is the first book that presents a systematic exposition of bounds for the spectra of various classes of linear operators in a Hilbert space. It is directed not only to specialists in functional analysis and linear algebra, but to anyone interested in various applications who has had at least a first year graduate level course in analysis. The functional analysis is developed as needed.

I was very fortunate to have had fruitful discussions with the late Professors I.S. Iohvidov and M.A. Krasnosel’skii, to whom I am very grateful for their interest in my investigations.

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# Table of Contents

## 1. Preliminaries
- 1.1 Vector and Matrix Norms ....................................... 1
- 1.2 Classes of Matrices ............................................ 2
- 1.3 Eigenvalues of Matrices ...................................... 3
- 1.4 Matrix-Valued Functions ...................................... 4
- 1.5 Contour Integrals ............................................. 5
- 1.6 Algebraic Equations .......................................... 6
- 1.7 The Triangular Representation of Matrices ................ 7
- 1.8 Notes ............................................................ 8

## 2. Norms of Matrix-Valued Functions
- 2.1 Estimates for the Euclidean Norm of the Resolvent ........ 11
- 2.2 Examples ....................................................... 13
- 2.3 Relations for Eigenvalues ................................... 14
- 2.4 An Auxiliary Inequality ...................................... 17
- 2.5 Euclidean Norms of Powers of Nilpotent Matrices ........ 18
- 2.6 Proof of Theorem 2.1.1 ...................................... 20
- 2.7 Estimates for the Norm of Analytic Matrix-Valued Functions .......... 21
- 2.8 Proof of Theorem 2.7.1 ...................................... 22
- 2.9 The First Multiplicative Representation of the Resolvent .... 24
- 2.10 The Second Multiplicative Representation of the Resolvent .... 27
- 2.11 The First Relation between Determinants and Resolvents .... 28
- 2.12 The Second Relation between Determinants and Resolvents .... 30
- 2.13 Proof of Theorem 2.12.1 .................................... 30
- 2.14 An Additional Estimate for Resolvents ..................... 32
- 2.15 Notes ........................................................... 33

## 3. Invertibility of Finite Matrices
- 3.1 Preliminary Results ........................................... 35
- 3.2 $l^p$-Norms of Powers of Nilpotent Matrices ................ 37

References .......................................................... 33
Table of Contents

3.3 Invertibility in the Norm $\|\cdot\|_p$ ($1 < p < \infty$) 39
3.4 Invertibility in the Norm $\|\cdot\|_\infty$ 40
3.5 Proof of Theorem 3.4.1 41
3.6 Positive Invertibility of Matrices 44
3.7 Positive Matrix-Valued Functions 45
3.8 Notes 47
References 47

4. Localization of Eigenvalues of Finite Matrices 49
4.1 Definitions and Preliminaries 49
4.2 Perturbations of Multiplicities and Matching Distance 50
4.3 Perturbations of Eigenvectors and Eigenprojectors 52
4.4 Perturbations of Matrices in the Euclidean Norm 53
4.5 Upper Bounds for Eigenvalues in Terms of the Euclidean Norm 56
4.6 Lower Bounds for the Spectral Radius 57
4.7 Additional Bounds for Eigenvalues 59
4.8 Proof of Theorem 4.7.1 60
4.9 Notes 62
References 62

5. Block Matrices and $\pi$-Triangular Matrices 65
5.1 Invertibility of Block Matrices 65
5.2 $\pi$-Triangular Matrices 67
5.3 Multiplicative Representation of Resolvents of $\pi$-Triangular Operators 69
5.4 Invertibility with Respect to a Chain of Projectors 70
5.5 Proof of Theorem 5.1.1 72
5.6 Notes 74
References 74

6. Norm Estimates for Functions of Compact Operators in a Hilbert Space 75
6.1 Bounded Operators in a Hilbert Space 75
6.2 Compact Operators in a Hilbert Space 77
6.3 Triangular Representations of Compact Operators 79
6.4 Resolvents of Hilbert-Schmidt Operators 83
6.5 Equalities for Eigenvalues of a Hilbert-Schmidt Operator 84
6.6 Operators Having Hilbert-Schmidt Powers 86
6.7 Resolvents of Neumilb-Schatten Operators 88
6.8 Proofs of Theorems 6.7.1 and 6.7.3 88
6.9 Regular Functions of Hilbert-Schmidt Operators 91
6.10 A Relation between Determinants and Resolvents 93
6.11 Notes 95
References 95
### 7. Functions of Non-compact Operators

#### 7.1 Terminology

- Page 97

#### 7.2 $P$-Triangular Operators

- Page 98

#### 7.3 Some Properties of Volterra Operators

- Page 99

#### 7.4 Powers of Volterra Operators

- Page 100

#### 7.5 Resolvents of $P$-Triangular Operators

- Page 101

#### 7.6 Triangular Representations of Quasi-Hermitian Operators

- Page 104

#### 7.7 Resolvents of Operators with Hilbert-Schmidt Hermitian Components

- Page 106

#### 7.8 Operators with the Property $A^p - (A^*)^p \in C_2$

- Page 107

#### 7.9 Resolvents of Operators with Neumann-Schatten Hermitian Components

- Page 108

#### 7.10 Regular Functions of Bounded Quasi-Hermitian Operators

- Page 109

#### 7.11 Proof of Theorem 7.10.1

- Page 110

#### 7.12 Regular Functions of Unbounded Operators

- Page 113

#### 7.13 Triangular Representations of Regular Functions

- Page 115

#### 7.14 Triangular Representations of Quasiunitary Operators

- Page 116

#### 7.15 Resolvents and Analytic Functions of Quasiunitary Operators

- Page 117

#### 7.16 Notes

- Page 120

#### References

- Page 120

### 8. Bounded Perturbations of Nonselfadjoint Operators

#### 8.1 Invertibility of Boundedly Perturbed $P$-Triangular Operators

- Page 123

#### 8.2 Resolvents of Boundedly Perturbed $P$-Triangular Operators

- Page 126

#### 8.3 Roots of Scalar Equations

- Page 127

#### 8.4 Spectral Variations

- Page 129

#### 8.5 Perturbations of Compact Operators

- Page 130

#### 8.6 Perturbations of Operators with Compact Hermitian Components

- Page 132

#### 8.7 Notes

- Page 134

#### References

- Page 134

### 9. Spectrum Localization of Nonself-adjoint Operators

#### 9.1 Invertibility Conditions

- Page 135

#### 9.2 Proofs of Theorems 9.1.1 and 9.1.3

- Page 137

#### 9.3 Resolvents of Quasinormal Operators

- Page 139

#### 9.4 Upper Bounds for Spectra

- Page 142

#### 9.5 Inner Bounds for Spectra

- Page 143

#### 9.6 Bounds for Spectra of Hilbert-Schmidt Operators

- Page 145

#### 9.7 Von Neumann-Schatten Operators

- Page 146

#### 9.8 Operators with Hilbert-Schmidt Hermitian Components

- Page 147

#### 9.9 Operators with Neumann-Schatten Hermitian Components

- Page 148
9.10 Notes ........................................... 149
References ........................................... 149

10. Multiplicative Representations of Resolvents 151
10.1 Operators with Finite Chains of Invariant Projectors .... 151
10.2 Complete Compact Operators .......................... 154
10.3 The Second Representation for Resolvents  
of Complete Compact Operators ........................ 156
10.4 Operators with Compact Inverse Ones .............. 157
10.5 Multiplicative Integrals ............................... 158
10.6 Resolvents of Volterra Operators ........................ 159
10.7 Resolvents of $P$-Triangular Operators ................ 159
10.8 Notes ........................................... 161
References ........................................... 161

11. Relatively $P$-Triangular Operators 163
11.1 Definitions and Preliminaries .......................... 163
11.2 Resolvents of Relatively $P$-Triangular Operators .... 165
11.3 Invertibility of Perturbed RPTO ........................ 166
11.4 Resolvents of Perturbed RPTO .......................... 167
11.5 Relative Spectral Variations ............................ 167
11.6 Operators with von Neumann-Schatten Relatively  
Nilpotent Parts ...................................... 168
11.7 Notes ........................................... 172
References ........................................... 172

12. Relatively Compact Perturbations of Normal Operators 173
12.1 Invertibility Conditions ............................... 173
12.2 Estimates for Resolvents ............................... 175
12.3 Bounds for the Spectrum ............................... 176
12.4 Operators with Relatively von Neumann - Schatten  
Off-diagonal Parts ................................... 177
12.5 Notes ........................................... 180
References ........................................... 180

13. Infinite Matrices in Hilbert Spaces  
and Differential Operators 181
13.1 Matrices with Compact off Diagonals .................... 181
13.2 Matrices with Relatively Compact Off-diagonals ....... 184
13.3 A Nonselfadjoint Differential Operator ................. 185
13.4 Integro-differential Operators .......................... 186
13.5 Notes ........................................... 187
References ........................................... 188
14.1 Scalar Integral Operators .................................. 189
14.2 Matrix Integral Operators with Relatively Small Kernels ... 191
14.3 Perturbations of Matrix Convolutions ....................... 193
14.4 Notes .................................................. 196
References ..................................................... 197

15. Operator Matrices 199
15.1 Invertibility Conditions .................................... 199
15.2 Bounds for the Spectrum ................................... 202
15.3 Operator Matrices with Normal Entries ...................... 204
15.4 Operator Matrices with Bounded off Diagonal Entries .... 205
15.5 Operator Matrices with Hilbert-Schmidt Diagonal Operators 207
15.6 Example .................................................. 209
15.7 Notes .................................................... 212
References ..................................................... 212

16. Hille - Tamarkin Integral Operators 215
16.1 Invertibility Conditions .................................... 215
16.2 Preliminaries ............................................. 217
16.3 Powers of Volterra Operators ................................ 219
16.4 Spectral Radius of a Hille - Tamarkin Operator .......... 221
16.5 Nonnegative Invertibility .................................. 222
16.6 Applications .............................................. 223
16.7 Notes .................................................... 226
References ..................................................... 226

17. Integral Operators in Space $L^\infty$ 227
17.1 Invertibility Conditions .................................... 227
17.2 Proof of Theorem 17.1.1 ................................... 228
17.3 The Spectral Radius ......................................... 230
17.4 Nonnegative Invertibility .................................. 231
17.5 Applications .............................................. 232
17.6 Notes .................................................... 234
References ..................................................... 234

18. Hille - Tamarkin Matrices 235
18.1 Invertibility Conditions .................................... 235
18.2 Proof of Theorem 18.1.1 ................................... 237
18.3 Localization of the Spectrum ................................ 238
18.4 Notes .................................................... 240
References ..................................................... 241