Mathematics and Visualization

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Preface

This book is entirely about triangulations. With emphasis on computational issues, we present the basic theory necessary to construct and manipulate triangulations. In particular, we make a tour through the theory behind the Delaunay triangulation, including algorithms and software issues. We also discuss various data structures used for the representation of triangulations. Throughout the book we relate the theory to selected applications, in particular surface construction, meshing and visualization.

The field of triangulation is part of the huge area of computational geometry, and over many years numerous books and articles have been written on the subject. Important results on triangulations have appeared in theoretical books and articles, mostly within the realm of computational geometry. However, many important results on triangulations have also been presented in publications within other research areas, where they have played and play an important role in solving specific scientific and applied problems. We will touch upon some of these areas in this book.

Triangulations, almost everywhere. The early development of triangulation comes from surveying and the art of constructing maps – cartography. Surveyors and cartographers used triangles as the basic geometric feature for calculating distances between points on the Earth’s surface and a position’s elevation above sea level.

Since the early development of computers an enormous effort has been put into theory, numerical methods and various algorithms for constructing and handling triangulations. This book gives most of the important results with respect to a family of triangulations that fulfill certain criteria. The most important type of triangulation we are investigating, the Delaunay triangulation, is named after the Russian mathematician Boris N. Delaunay, who first described this important triangulation scheme in 1934. During recent decades, advances in computer hardware and software have also brought triangulation
technology into many new areas of application. Together with some new results, we cover the major achievements of Delaunay triangulation methods and some important applications of such triangulations.

Triangle-based surfaces, or surfaces represented over triangulations, are used in a wide range of applications. They are to be found in computer aided design (CAD) systems in the automotive industry, and they are used for defining meshes and geometry needed in systems for simulating processes and phenomena based on finite element methods (FEM). Moreover, they are extensively used in systems for representing the geometry of geological structures, and in medical applications triangulations are used for representing the anatomy of the human body. Triangulations are also used in geographical information systems (GIS), mainly for the purpose of representing parts of the Earth’s surface, and most of the early results on the subject were developed by practical cartographers for creating maps. Finally, triangulations have become one of the main features in visualization and computer graphics. In particular, the game industry has pushed the development of graphics hardware in order to obtain real-time visualization of as huge data sets as possible. Triangulations have been of particular interest to the developers of graphics hardware, mainly because of their ultimate simplicity – a triangulation is a collection of triangles, and a triangle is given by three points in space. Three points are the minimum number of points needed to represent a piece of a surface in space.

Measurement data, in many cases also referred to as scattered data, are results of spatial measurements taken from some physical body, e.g. the Earth’s surface. Data are collected from satellite images, aeroplane images, the global positioning system (GPS) and other types of equipment for measuring position. Geological data most often comes from seismic surveys and well measurements. In medicine, data can be extracted from ultrasound images and images created by magnetic resonance imaging (MRI). A common challenge in all these areas is to construct the underlying geometry of the measured object, and many systems today use triangles and triangulations for this purpose.

Although the theory presented in this book is general and we are aware of the wide variety of applications which use triangulations, we use surface construction and meshing as the main examples throughout the book. In particular, we are interested in the construction and manipulation of triangulations based on some suitable set of measured data.

*Teaching triangulations.* This book is based on lecture notes from a course on triangulation given at graduate level at the Department of Informatics at the University of Oslo. The course is given over one semester and tailored to be one third of a full-time student’s workload during the semester. The semester typically starts mid August with oral exam sometime during December. The
course is given over 12–14 double lectures and a number of programming exercises based on the companion software. Except for the final chapter on software, the organization of the chapters is based on the chronology of our lectures. The students are introduced to the companion software when needed as programming exercises are given throughout the course.

**How to read this book.** Although our advice is to follow the chronology of the chapters, the reader can also follow other paths through this book.

- An introduction to triangles and triangulations and the basic theory of Delaunay triangulations are given in Chapters 1, 3 and 4. In Chapter 3 we define the Delaunay triangulation and in Chapter 4 we discuss the most important algorithms for constructing the Delaunay triangulation. In Chapter 1 we give a brief introduction to triangles and triangulations and give some necessary basic properties of triangulations.

- Chapters 5, 6 and 7 contain three types of triangulations based on the theory of Delaunay triangulation. All three chapters are heavily based on the basic theory given in Chapters 3 and 4. In Chapter 5, we describe methods for constructing triangulations that depend on the shape or the behavior of the given data. In Chapter 6, we introduce constraints in the triangulation itself by keeping edges of triangles fixed, and in Chapter 7, we construct triangle-based meshes suited for finite element calculations.

- In Chapter 8, we use triangulations as a basis for constructing surfaces from huge data sets. This chapter can be read either separately or after a brief look at Chapter 1.

- Chapters 2 and 9 are directed towards implementation issues. In Chapter 2, we discuss data structures for representing triangulations and Chapter 9 is dedicated to generic software components for constructing and manipulating triangulations.

Another possible path through the book is as follows: start with Chapters 1, 3 and 4, continue with Chapters 2 and 9, then read Chapter 8, before you end the course with a selection from the contents of Chapters 5, 6 and 7.

**Related topics.** As pointed out above, this book is based on lecture notes for a graduate level course at the Department of Informatics at the University of Oslo. In order to give the students a more visual understanding of triangulations, we usually add some lectures on visualization and graphics, including a short introduction to OpenGL [1, 89].

We have decided not to give a detailed introduction to simplification and refinement of triangular meshes, except for the specific mesh generation scheme discussed in Chapter 7. However, insertion and deletion of points in triangulations are covered by the general theory and can therefore be regarded as a basis for both simplification and refinement. For those who are particularly interested in simplification of triangular meshes, see for example [26].
In [19], de Berg, van Kreveld, Overmars and Schwarzkopf give a thorough introduction to computational geometry where they also cover important aspects of triangulations and algorithms for handling triangular meshes. In [26], H. Edelsbrunner covers topics in geometry and topology applied to grid and mesh generation where the theory behind Delaunay triangulations and Voronoi diagrams is also discussed. In addition to surface simplification as mentioned above, Edelsbrunner also covers tetrahedral meshes, which are natural extensions of triangulations to three dimensions.

Software companion. An important feature of this book is the companion software, the Triangulation Template Library (TTL). TTL is open source software and can be downloaded from www.simula.no/ogl/ttl. The main ideas behind TTL and description of its functionality are given in the last chapter of this book. We present a generic programming philosophy for triangulations with a clear separation of algorithms and data structures. Since TTL is frequently extended and changed, we have decided not to exploit TTL in too much detail in this book. Important software issues and implementation aspects can be found on the TTL homepage.

Acknowledgement. As mentioned above, this book is based on lecture notes, and first of all we would like to thank all the students who have attended the course and have given us continuous feedback on the contents, outline and details as the lecture notes have evolved. In particular, we wish to thank students Thomas Elboth, Øystein Aanrud, Stein Grongstad, Tom F. Blenning Klaussen, Siri Øyen Larsen, Per-Idar Evensen and Philip Bruvold for their detailed feedback. In addition, Per-Idar Evensen has done excellent work on testing the software as a part of his Master thesis. A special thank you goes to Thomas Sevaldrud who has provided us with valuable comments on the contents, for the production of computer graphics examples and for extensive use and feedback on the software. We wish to thank Hans Petter Langtangen, Kjell Kjenstad, Kyrre Strøm and Martin Reimers for their comments and valuable feedback on specific parts of the manuscript. Finally, we would like to thank Professor Jonathan Shewchuk at Berkeley for letting us use his brilliant illustrations on Delaunay meshing in Chapter 7. We also thank SINTEF Applied Mathematics for supporting the development of the Triangulation Template Library (TTL). The cover picture is taken from the commercial flight simulator Silent Wings (www.silentwings.no) that uses our companion software TTL for the construction of the triangle-based terrain surface.

Norway
May 2006

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