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Homological Methods, Representation Theory, and Cluster Algebras
This volume includes six mini-courses delivered at the 2016 CIMPA (Centre International de Mathématiques Pures et Appliquées) research school held in the Universidad Nacional de Mar del Plata, Mar del Plata, Argentina, from the 7th to the 18th of March 2016. More than 80 mathematicians and students from a dozen countries participated in the event.

This research school was dedicated to the founder of the Argentinian research group in representation theory of algebras, Dr. M.I. Platzeck, on the occasion of her 70th birthday. It was devoted to interactions between representation theory, homological algebra and the new ever-expanding theory of cluster algebras. While homological algebra has always been present as one of the main tools in the study of finite dimensional algebras, the more recent strong connection with cluster algebras quickly established itself as one of the important features of the mathematical landscape. This connection has been fruitful to both areas, representation theory provided a categorification of cluster algebras, while the study of cluster algebras provided representation theory with new objects of study like tilting in the cluster category. This volume stands as a partial testimony to this new and welcome development.

The six courses presented at the research school were organised as follows. During the first week the more elementary courses were delivered (in this volume, the courses “Introduction to the Representation Theory of Finite-Dimensional Algebras: The Functorial Approach,” “Auslander–Reiten Theory for Finite-Dimensional Algebras” and “Cluster Algebras from Surfaces”), the first two of which form the basis of modern-day representation theory and the third one an introductory course on an important class of cluster algebras. The more advanced courses, which concentrate on connections between representation theory and cluster algebras, took place during the second week (in this volume, the courses “Cluster Characters,” “A Course on Cluster-Tilted Algebras” and “Brauer Graph Algebras”). We would like to express our gratitude to the authors who submitted contributions and to the referees for their assistance.

The courses in this volume are addressed to graduate students or young researchers with some previous knowledge of noncommutative algebra or homological algebra. This volume will also be of interest to any mathematician who is not a specialist of the topics presented here and would like to access this fast-developing field. Because interactions between topics of the research school can only increase, and the courses presented reflect
the diversity as well as the rich activity of the groups working in the area, we hope that this volume will be useful to its readers.

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