Applications of Measure Theory to Statistics
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About the Author

Prof. Gogi Pantsulaia graduated in Mathematics from the Iv. Javakhishvili Tbilisi State University (Georgia) in 1982, and received his Ph.D. in mathematics at the Institute of Mathematics of the Ukrainian Academy of Sciences (Ukraine) in 1985. In 2003 he received his degree of doctor of physics and mathematics from the I. Vekua Institute of Applied Mathematics of the Iv. Javakhishvili Tbilisi State University (Georgia). He is a full professor in the Department of Mathematics of the Georgian Technical University (Georgia). He has participated in more than 10 research projects and is the author of four books and more than 75 papers. He has also supervised several doctoral theses. His current research interests include set theory, measure theory, probability theory, and mathematical statistics.
Introduction

As Robinson and Wainer [RW] have observed in the almost 300 years since its introduction by Arbuthnot [A], null hypothesis significance testing (NHST) has become an important tool for working scientists. In the early 20th century, the founders of modern statistics (R.A. Fisher, Jerzy Neyman, and Egon Pearson) showed how to apply this tool in widely varying circumstances, often in agriculture, that were almost all very far afield from Dr. Arbuthnot’s noble attempt to prove the existence of God.

In the process of applying NHST, however, confusion has often arisen among practitioners and statisticians, giving rise to philosophical criticisms of hypothesis testing (see, for example, [Ch, LMS, K, MZ, MH, O], which cite 300–400 primary references). These criticisms reflected a general point of view that the theory of mathematical statistics and the results of testing are inconsistent in many situations and that the typical null hypothesis is almost always false. With the advantage of increasing use, practitioner’s eyes became accustomed to the darker reality and the shortcomings of NHST became more apparent.

Below we cite comments concerning the criticism of NHST from the joint work of J.L. Rodgers and D.C. Rowe [RR]:

The methodological literature contains cogent criticism of null hypothesis significance testing (NHST; e.g., Cohen, 1994; Rozeboom, 1960; Schmidt, 1996), including the extreme position that NHST has been discredited and never makes a positive contribution (Schmidt Hunter, 1997, p. 37). Some have called for NHST to be outlawed, with enforcement charged to journal editors (see discussion in Shrout, 1997). The American Psychological Association (APA) Board of Scientific Affairs commissioned a task force on statistical inference, a committee of talented methodological scholars, to consider the situation. Their conclusion (Wilkinson Task Force on Statistical Inference, APA Science Directorate, 1999) was to temper the stridency of the anti-NHST movement:

Some had hoped that this task force would vote to recommend an outright ban on the use of significance tests in psychology journals. . . . the task force hopes instead that this report will induce editors, reviewers, and authors to recognize practices that institutionalize the thoughtless application of statistical methods. (pp. 602–603).
Most thoughtful methodologists seem to conclude that NHST is a statistical tool that handles a certain type of question (e.g., Abelson, 1997; Muliak, Raju, Harshman, 1997; Wainer, 1999). As with any tool, problems can arise with misuse, and other tools can substitute for or complement NHST. However, the committee and many others concluded that NHST should remain part of the methodologists toolbox, along with confidence intervals, effect sizes, graphical analysis, and so on. We were reminded of this recent NHST controversy when we read Roberts and Pashler’s (2000) article, which criticized the evaluation of how well a model fits empirical data in the development of psychological and social science theories. Like the critics of NHST, Roberts and Pashler had no difficulty identifying examples in which model-fitting methodology has been misused. Like the critics of NHST, they also pointed to methods to evaluate models that can substitute or complement the evaluation of goodness-of-fit procedures. Like the critics of NHST, they substantially overstated their case in their criticism of using good fits to support theories (p. 365).

On the one hand, the reexamination of the viability of NHST was described by Anderson, Burnham, and Thompson (2000), who showed that over the past 60 years an increasing number of articles have questioned the utility of NHST.

On the other hand, it is revealing that Thompson’s database, over the same time period, showed a concomitant increase in the number of articles defending the utility of NHST.

It is obvious that this phenomenon can formally be explained as follows:

- An application of NHST has a supporter (or an oppositionist) if the associated statistical test is “objective” (or “subjective”).

One of the purposes of this book consists in putting into notions “objective” and “subjective” reasonable mathematical senses and in providing this simple explanation with a strong mathematical base.

It seems worthwhile also to use “subjective” statistical tests to try to construct new “objective” statistical tests under which NHST remains a viable tool. Here we will present a methodology for resolution of this problem under some restrictions introduced by Pantsulaia and Kintsurashvili [PK2].

We are not going to consider in detail all relevant issues. Instead we shall focus our attention on a certain confusion which is described in the works of Nunnally [N] and Cohen [Coh].

In 1960, Nunnally [N] noticed that in many standard statistical tests null hypotheses are always rejected and observed: “If the decisions are based on convention they are termed arbitrary or mindless while those not so based may be termed subjective. To minimize type II errors, large samples are recommended. In psychology practically all null hypotheses are claimed to be false for sufficiently large samples so … it is usually nonsensical to perform an experiment with the sole aim of rejecting the null hypothesis”.

In 1994, Cohen [Coh] noticed some gaps between the theory of mathematical statistics and the results of testing and observed: “… Don’t look for a magic alternative to NHST [null hypothesis significance testing] … It does not exist.”
Here the following question naturally arises:

- **Whether can be explained Jacob Cohen and Jum Nunnally above mentioned observations?**

Estimating a useful signal for a linear one-dimensional stochastic system, we plan to demonstrate the validity of Cohen and Nunnally’s predictions for a certain standard hypothesis testing in terms of infinite samples such that the sum of errors of I and II types is equal to zero (we refer to such tests as tests with a maximal reliability). Note that working with infinite samples is a natural requirement because a definition of consistent estimates can not be given without infinite samples. Further, we plan to explain why a null hypothesis is claimed to be false for “almost every” (in the sense of [HSY]) infinite sample.

Another goal of the present book is an application of the approach of “almost every” (in the sense of [HSY]) in studying structures of domains of some infinite sample statistics and in explaining why the null hypothesis is rejected for “almost every” (in the sense of [HSY]) infinite sample by the associated NHST with a maximal reliability.

In order to explain the large gap between the theory of mathematical statistics and the results of hypothesis testing, by using the technique of Haar null sets in the space of infinite samples, we introduce an essentially new approach which naturally divides the class of all consistent infinite sample estimates of a useful signal in the linear one-dimensional stochastic model into disjoint classes of subjective and objective estimates. Following this approach, each consistent infinite sample estimate has to pass a theoretical test on objectivity. This means that theoretical statisticians should expend much effort in carrying out such a certification exam for each consistent infinite-sample estimation.

Correspondingly, we have the following three objectives:

i. To introduce a new approach which naturally divides the class of all consistent estimates of an unknown parameter in a Polish group into disjoint classes of subjective and objective estimates.

ii. To construct tests on objectivity for consistent estimations of an unknown parameter in a Polish group.

iii. To explain of the main requirement why each consistent infinite-sample estimation must pass the certification exam on objectivity.

This book is devoted to the mathematical development of the first two items. We also briefly discuss a few interesting mathematical points concerning item (iii) (While there is a rich family of NHTSs whose corresponding statistical tests are consistent, to date we have no information regarding their objectivity).

The book comprises six chapters, as outlined below:

Chapter 1 demonstrates that the technique for numerical calculation of some one-dimensional improper Riemann integrals is similar to the technique given by Weyl’s [W] celebrated theorem for continuous functions on \([0, 1]\).

In Sect. 1.2 we consider some auxiliary notions and facts from the theory of uniformly distributed sequences on the interval \([0, 1]\). In Sect. 1.3 we present the
proof of a certain modification of Kolmogorov’s Strong Law of Large Numbers and
the Glivenko-Cantelli theorem. In Sect. 1.4 we give an extension of the main result
of Baxa and Schoiβengeier [BS] for calculation of some improper one-dimensional
Riemann integrals by use of uniformly distributed sequences.

Chapter 2 presents a concept of infinite-dimensional Monte-Carlo integration
developed by Pantsulaia [P6].

In Sect. 2.2, in terms of the “Lebesgue measure” λ [B1], we consider concepts
of the uniform distribution and the Riemann integrability in infinite-dimensional
rectangles in $R^\infty$ and prove infinite-dimensional versions of the famous results
of Lebesgue [N1] and Weyl [W], respectively. In this section we show that if $(\alpha_{n(k)})_{n\in N}$
is an infinite sequence of different integer numbers for every $k \in N$, then a set of all
sequences $(x_k)_{k\in N}$ in $R^\infty$ for which a sequence of increasing sets $(Y_n((x_k)_{k\in N}))_{n\in N}$
defined by

$$Y_n((x_k)_{k\in N}) = \prod_{k=1}^{n} \left( \bigcup_{j=1}^{n} \{ < a_{j(k)} x_k > (b_k - a_k) \} + a_k \right) \times \prod_{k \in N \setminus \{1, \ldots, n\}} \{ a_k \}$$

is not λ-uniformly distributed on the $\prod_{k \in N} \{ a_k, b_k \}$ is of λ measure zero and, hence shy in $R^\infty$, where $\langle \cdot \rangle$ denotes the fractional part of the real number.

In Sect. 2.3, a Monte-Carlo algorithm for estimating the value of infinite-dimensional Riemann integrals over infinite-dimensional rectangles in $R^\infty$ described by Pantsulaia [P6] is presented. Further, we introduce Riemann integrability for real-valued functions with respect to product measures in $R^\infty$ and give some sufficient conditions under which a real-valued function of infinitely many real variables is Riemann integrable. We describe a Monte-Carlo algorithm for computing of infinite-dimensional Riemann integrals for such functions.

In Sect. 2.4, we consider some interesting applications of Monte-Carlo algo-
rithms for computing of the infinite-dimensional Riemann integrals described in Sect. 2.3.

Chapter 3 is devoted to study of the structure of the set of all sequences uniformly distributed in $[-1/2, 1/2]$. Pantsulaia [P5] has shown that μ-almost every element of $R^\infty$ is uniformly distributed in $[-1/2, 1/2]$, where μ denotes Moore-Yamasaki-Kharazishvili measure in $R^\infty$ for which $\mu([-1/2, 1/2]^\infty) = 1$. In Sect. 3.3 we prove that the set $D$ of all real-valued sequences uniformly distributed in $[-1/2, 1/2]$ is shy in $R^N$. In Sect. 3.4, we demonstrate that in the Solovay model [So1] the set $F$ of all sequences uniformly distributed modulo 1 in $[-1/2, 1/2]$ is prevalent set [HSY] in $R^N$.

Chapter 4 contains a brief description of Yamasaki’s [Y] remarkable investigation
(1980) of the relationship between Moore-Yamasaki-Kharazishvili type measures
and infinite powers of Borel diffused probability measures on $R$. More precisely,
Yamasaki’s proof is given that no infinite power of the Borel probability measure
with a strictly positive density function on $R$ has an equivalent Moore-Yamasaki-
Kharazishvili type measure. A certain modification of Yamasaki’s example is used
for the construction of such a Moore-Yamasaki-Kharazishvili type measure that is
equivalent to the product of a certain infinite family of Borel probability measures with a strictly positive density function on $R$. By virtue of the properties of real-valued sequences equidistributed on the real axis, it is demonstrated that an arbitrary family of infinite powers of Borel diffused probability measures with strictly positive density functions on $R$ is strongly separated and, accordingly, has an infinite-sample well-founded estimator of the unknown distribution function. This extends the main result established in [ZPS].

The last two chapters of the book present applications of the theories of Haar null sets and of uniformly distributed sequences in $[0, 1]$ to statistics.

In Chap. 5, by using the notion of a Haar ambivalent set introduced by Balka, Buczolich and Elekes [BBE], essentially new classes of statistical structures having objective and strong objective estimates of unknown parameters are introduced in a Polish non-locally-compact group admitting an invariant metric and relations between them are studied. An example of a weakly separated statistical structure is constructed for which a question asking “whether there exists a consistent estimate of an unknown parameter” is not solvable in the theory $(ZF) \& (DC)$. A question asking “whether there exists an objective consistent estimate of an unknown parameter for any statistical structure in a non-locally compact Polish group with an invariant metric when subjective one exists” is answered positively in [KKP] when there exists at least one such a parameter the pre-image of which under this subjective estimate is a prevalent set. This construction essentially uses the rather recent celebrated result of Solecki [So2] concerning the partition of a non-locally compact Polish group into a continuous family of pairwise disjoint Haar ambivalents. These results are extensions of recent results of Pantsulaia and Kintsurashvili [PK2]. Some examples of objective and strong objective consistent estimates in a compact Polish group $\{0; 1\}^N$ are also considered in this chapter. At the end of the chapter we present a certain claim for theoretical statisticians in which each consistent estimation with domain in a non-locally compact Polish group equipped with an invariant metric must pass the certification exam on objectivity prior to its practical application and give some recommendations.

In Chap. 6, the notion of Haar null set firstly introduced by Christensen [Ch1] in 1973 and reintroduced in 1992 in the context of dynamical systems by Hunt, Sauer and Yorke [HSY] is used in studying structures of domains of some infinite sample statistics (for example, of an infinite sample average) and in explaining why the null hypothesis is rejected for “almost every” infinite sample by hypothesis testing with maximal reliability.
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Introduction

