Part IX
Special Topics in Harmonic Analysis
Part IX consists of two chapters, that are as distinctive as their FFT authors are distinguished. GILBERT STRANG spoke at FFT 2012 and ROBERT S. STRICHARTZ spoke at FFT 2002, our first FFT; and their two chapters comprise Part I.

STRANG goes deeply into the analysis of the well-known factorization $A = LPU$ of an invertible $n \times n$ matrix $A$, where $L$ is lower triangular, $U$ is upper triangular, and $P$ is a unique permutation matrix. Recall that an $n \times n$ permutation matrix is defined by the condition that it has the entry 1 in each row and in each column and is 0 for all other entries. Natural adjustments of $A = LPU$ lead to the Wiener–Hopf form $A = UPL$ and the Bruhat decomposition $A = U_1 \pi U_2$.

The Wiener–Hopf form is a natural matricial formation of Wiener–Hopf’s method to solve systems of integral equations, as well as certain systems of partial differential equations arising in mathematical physics. The basic technique of Wiener and Hopf comes down to defining two complex functions $\Phi_+$ and $\Phi_-$, where $\Phi_+$ (respectively, $\Phi_-$) is analytic in the upper (respectively lower) half-plane; and $U$ (respectively, $L$) is the analogue of $\Phi_+$ (respectively, $\Phi_-$).

The factorization, $A = LPU$, for an $n \times n$ matrix $A$, can be viewed classically in terms of an interpretation of the Gaussian elimination method of solving $n$ linear equations in $n$ unknowns. Strang’s main results deal with elimination on banded doubly infinite matrices. His insights are pivotal (sic) and magisterial, and his examples are truly enlightening.

BELLO, LI, AND STRICHARTZ outline a Hodge-de Rham theory of $k$-forms ($k = 0, 1, 2$) on a Sierpiński carpet. A wonderful aspect of this paper is that Strichartz’ two co-authors were undergraduates at the time of the research, and were part of Strichartz’ now famous Research Experience for Undergraduates (REU) program sponsored by the National Science Foundation (NSF).

The Sierpiński carpet is a fractal and the authors approximate it by a sequence of graphs, use classical Hodge-de Rham theory on each graph, and take the limit. The interplay of mathematical and computational tools is labyrinthine and fascinating. The Sierpiński carpet, SC, itself is defined in terms of similarity maps, $F_j$, of contraction ratio $1/3$. It is the analogue in $\mathbb{R}^2$ of the $1/3$-Cantor set. However, SC is a connected set, as well as being compact with Lebesgue measure 0. Further, it cuts the plane into infinitely many disjoint parts. There is a natural way to associate a sequence of graphs, $\Gamma_m$, to $\{F_j\}_{j=1}^\infty$, and ultimately to define the associated de Rham complex for each $m$. Then, there is extensive experimentation and the definition of 0-, 1-, and 2-forms on the SC and the analysis of the corresponding Laplacians. For example, the Laplacian $-\Delta_0^{(m)}$ for the 0-form is exactly the graph Laplacian of $\Gamma_m$ with specified weights on the vertices and edges—all very heady-stuff, quite like de Rham’s challenges to the lofty Alps.

Part of analysis, some of which is open-ended and important, is the characterization of the spectra of the Laplacian on $k$-forms. Some of the fascination is the tantalizing possible relationship with the role of Laplacians on graphs with regard to current interest in dimension reduction as related to “big data.”