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Classical and Stochastic Laplacian Growth

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Preface

One of the most influential works in fluid dynamics at the edge of the XIXth and XXth centuries was a series of papers, see, e.g., [265], written by Henry Selby Hele-Shaw between 1897 and 1899. This was a time of impetuous development of several fundamental branches of natural sciences. The following few citations related to the present monograph might be sufficient to impress an inquiring minded reader: Reynolds’ description [466] of the turbulence phenomenon at higher velocities in 1873–1883; Korteweg and de Vries’ description [320] of the ‘solitary wave’ in 1895, discovered by the Scottish engineer Russell about half a century earlier; and of course, we must mention the impressive developments in the theory of relativity and quantum physics, which used the elegant mathematical formulation of electrodynamics given by Maxwell [382], both as a benchmark for more unified theories (which emerged later in the XXth century as gauge theories), and as a guiding mathematical structure (reflected nowadays in the extensive use of principal bundles and central extensions in theoretical physics).

Hele-Shaw (1854–1941) was one of the most prominent engineering researchers of his time, a pioneer of technical education, a great organizer, president of several engineering societies, including the Royal Institution of Mechanical Engineers, Fellow of the Royal Society, and sadly, an example of one of the many undeservedly forgotten great names in Science and Engineering. In his original works, he first described his famous cell that became a subject of deep investigation only more than 50 years later. A Hele-Shaw cell is a device for investigating two-dimensional flow of a viscous fluid in a narrow gap between two parallel plates. This cell is the simplest system in which multi-dimensional convection is present. Probably the most important characteristic of flows in such a cell is that when the Reynolds number based on gap width is sufficiently small, the Navier–Stokes equations averaged over the gap reduce to a linear relation similar to Darcy’s law and then to a Laplace equation for pressure. Different driving mechanisms can be considered, such as surface tension or external forces (suction, injection). Through the similarity in the governing equations, Hele-Shaw flows are particularly useful for visualization of saturated flows in porous media, assuming they are slow enough to be governed by Darcy’s law. Nowadays, the principle of the Hele-Shaw cell is used as a powerful tool for modelling growth phenomena in several fields of natural sciences and engineering, in particular, condensed-matter physics, material science, crystal growth and, of course, fluid mechanics. But Hele-Shaw
is known not only for his Stream-line Flow Methods (1896–1900) in which this
cell plays a fundamental role. Two other of Hele-Shaw’s great inventions are his
Friction Clutch (1905), an early version of multi-plate wet clutch, and his Auto-
matic Variable-Pitch Propeller (1924), jointly with T. Beacham. In fact, the full
list of his inventions is much longer and comprises 82 patents. Sir George Gabriel
Stokes wrote about the Hele-Shaw cell: ‘Hele-Shaw’s experiments afford a com-
plete graphical solution, experimentally obtained, of a problem which from its
complexity baffles mathematicians except in a few simple cases’. Stokes mentions
also Hele-Shaw’s experiments in his letter to Lord Kelvin from September 7, 1898:
‘Hele-Shaw has some beautiful photographs, very interesting to you and me. By
means of a thin stratum of viscous liquid between close glass walls, flowing past an
interruption in the film, you can realise experimentally the theoretical stream lines
in two dimensions in a perfect fluid flowing round a body represented in section
by the obstacle’ (see [532]).

Since the original works of Hele-Shaw appeared and a mathematical model
of Hele-Shaw flow was formulated in the famous monograph by Lamb [341], many
interesting and exciting developments have occurred. The century-long develop-
ment connecting the original Hele-Shaw experiments, the conformal mapping for-
mulation of the Hele-Shaw flow by Pelageya Yakovlevna Polubarinova-Kochina
(1899–1999) and Lev Aleksandrovich Galin (1912–1981) [438, 439, 199], and the
modern treatment of the Hele-Shaw evolution based on integrable systems and
on the general theory of plane contour motion, was marked by several important
contributions by individuals and groups.

The main idea of Polubarinova-Kochina and Galin was to apply the Riemann
mapping from an appropriate canonical domain (the unit disk in most situations)
onto the phase domain in order to parameterize the free boundary. The evolu-
tion equation for this map, named after its creators, allows us to construct many
explicit solutions and to apply methods of conformal analysis and geometric func-
tion theory to investigate Hele-Shaw flows. In particular, solutions to this equa-
tion in the case of advancing fluid give subordination chains of simply connected
domains which have been studied for a long time in the theory of univalent func-
tions. The Löwner–Kufarev equation [328], [362] plays a central role in this study
(Charles Loewner or Karel Löwner originally in Czech, 1893–1968; Pavel Par-
equations, having some evident geometric connections, are of somewhat different
nature. While the evolution of the Laplacian growth given by the Polubarinova–
Galin equation is completely defined by the initial conditions, the Löwner–Kufarev
evolution depends also on an arbitrary control function. The Polubarinova–Galin
equation is essentially non-linear and the corresponding subordination chains are
of rather complicated nature. Interestingly, it was Kufarev [330], [331] who antic-
pipated further results in viscous fingering in 1948 by means of this equation.

Among other remarkable contributions we distinguish the discovery of the
viscous fingering phenomenon by Sir Geoffrey Ingram Taylor (1886–1975) and
Philip Geoffrey Saffman (1931–2008) [488, 489], and the discovery, by Stanley Richardson (1943–2008) [470], of a complete set of integrals for the Hele-Shaw evolution, namely the harmonic moments. Contributions made by scientists from Great Britain (D. Crowdy, L.J. Cummings, C.M. Elliott, S.D. Howison, J.R. King, J.R. Ockendon, S. Richardson) are to be emphasized. They have substantially developed the complex variable approach and actually converted the Hele-Shaw problem into a modern challenging branch of applied mathematics. An even more recent mathematical physics perspective, through integrable systems in particular, allows us to look at Hele-Shaw evolution as at a general contour dynamics in the plane embedded into a dispersionless Toda hierarchy. This approach is due mainly to I. Krichever, A. Marshakov, M. Mineev-Weinstein, P. Wiegmann, A. Zabrodin among others.

The first monograph treatment [249] of Hele-Shaw appeared in 2006 and covered mostly the classical period of development of this area (a related monograph is [560]). The last decade has been marked by a burst of interest in Laplacian growth (another name for the Hele-Shaw free boundary problem used in mathematical physics literature), caused in particular by related statistical physics models, e.g., Diffusion Limited Aggregation. Several new methods, such as integrable systems and random matrices, have been employed to treat problems in Laplacian growth. Therefore, a revision of the book seemed necessary in order to give a broader and more comprehensive survey of the current status of this field, whose impetuous growth has resulted in this new text, with three authors. While Chapters 6 through 9 are entirely new (except for the first section in Chapter 7), the main ideas of [249] are also present, in particular in the first half of the introduction (Chapter 1, Sections 1–7), Sections 2, 5, and 6 of Chapter 2 (and an extended Section 1), Sections 1–2 of Chapter 3 (and partly Sections 3 and 5), Sections 1, 3, 5, and 7 of Chapter 4, and finally Sections 1–4 of Chapter 5.

In the present monograph, we aim at giving a presentation of recent and new ideas that arise from the problems of planar fluid dynamics and which are interesting from the point of view of geometric function theory, potential theory, and mathematical physics. In particular, we are concerned with geometric problems for Laplacian growth, its stochastic formulation and its treatment from the viewpoint of integrable systems and random matrices. Ultimately, we see the interaction between several branches of complex, potential analysis, mathematical physics and planar fluid mechanics.

For most parts of this book we assume the background provided by graduate courses in real and complex analysis, in particular the theory of conformal mappings, and some basic notions of fluid mechanics. We also make some historical remarks concerning the scientists who have contributed to the topic. We have tried to keep the book as self-contained as possible.

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