Operator Theory: Advances and Applications
Volume 234

Founded in 1979 by Israel Gohberg

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Separable Type Representations of Matrices and Fast Algorithms

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Preface

Our interest in structured matrices was inspired by the outstanding mathematician Professor Israel Gohberg, our older friend, colleague and teacher. His ideas, projects and our joint results lie at the basis of this book. His supervision and participation were crucial in the preparation of the manuscript. On 12 October 2009 Israel Gohberg passed away and we finished the book alone. Our work is devoted to his blessed memory.

October 31, 2012  Yuli Eidelman, Iulian Haimovici
Introduction

The book. The majority of the basic algorithms of computations with matrices are expressed via the entries of the matrices and are not taking into account the individual properties or the specific structure of these matrices. This often results in a unjustified high complexity of the algorithms.

For instance, the multiplication of two matrices of order $N$ via the entries of the matrices requires in general $N^3$ operations. For many classes of structured matrices this complexity can be reduced by an appropriate presentation of the factors and the product as well as the algorithm. For this purpose we have to represent the matrices and the algorithm not in terms of the entries of the matrices, but in terms of other parameters (generators) which are essentially involved in the description of the structure of these matrices. For matrices of the form

$$A = \{a_{ik}\}_{i,k=1}^{N}, \quad a_{ik} = x_i^T y_k, \quad i, k = 1, 2, \ldots, N$$

with $x_i, y_k \in \mathbb{C}^n$, $n \ll N$, which are often called separable matrices, the natural parameters (generators) are the $n$-dimensional vectors $x_i, y_k (i, k = 1, \ldots, N)$. The computations for matrices of this form in terms of the natural parameters are of a much lower complexity. So for the product of two such matrices, $A$ and

$$B = \{b_{kj}\}_{k,j=1}^{N}, \quad b_{kj} = v_k^T u_j, \quad k, j = 1, 2, \ldots, N,$$

we get $C = AB = \{c_{ij}\}_{i,j=1}^{N}$ with

$$c_{ij} = \sum_{k=1}^{N} x_i^T y_k v_k^T u_j = x_i^T \left( \sum_{k=1}^{N} y_k v_k^T \right) u_j.$$

Hence the product $C$ is a matrix with separable generators $x_i$ and

$$w_j = \left( \sum_{k=1}^{N} y_k v_k^T \right) u_j.$$

To compute the sum $a = \sum_{k=1}^{N} y_k v_k^T$ one requires $Nn^2$ operations and the products $w_j = au_j (j = 1, \ldots, N)$ cost $Nn^2$ operations. Thus for the multiplication of two
matrices in separable form one needs only $2n^2N$ operations. If $n$ is fixed, the complexity is asymptotically equal to $O(N)$. A similar situation appears also for the inversion of matrices of this type.

This book contains a systematic theoretical and computational study of several types of generalizations of separable matrices. It is related to semiseparable, quasiseparable, band and companion representations of matrices. For them their natural parameters, called generators, are analyzed and algorithms are expressed in terms of generators. Connections between matrices and boundary value problems for discrete systems play an important role. The book is focused on algorithms of multiplication, inversion and description of eigenstructure of matrices. A large number of illustrations are provided in the text. The book consists of eight parts.

**Description of parts.** The first part is mainly of a theoretical character. Here we introduce the notions of quasiseparable and semiseparable structure. These notions are illustrated on some well-known examples of tridiagonal matrices, band matrices, diagonal plus semiseparable matrices, scalar and block companion matrices. We derive various properties of quasiseparable and semiseparable structure which are used in the sequel. An essential part of the material concerns the minimal rank completion problem.

The second part is devoted to completion to Green matrices and to unitary matrices and also to the completion of mutually inverse matrices.

Discrete systems with boundary conditions allow to present a transparent description of various algorithms which is started in the third part. We begin the presentation of algorithms with multiplication by vectors and then with algorithms which are based on some well-known inversion formulas via quasiseparable structure. An essential role in this part plays the interplay between the quasiseparable structure and discrete-time varying linear systems with boundary conditions.

The fourth part contains factorization and inversion algorithms for matrices via quasiseparable and semiseparable structure. We present the LDU factorization and inversion algorithms for strongly regular matrices. Algorithms of this type are extended to arbitrary matrices with quasiseparable representations of the first order. In the last chapter algorithms for the QR factorization and the QR based solver for linear algebraic systems are presented.

The second volume is divided into Parts V–VIII. The titles are as follows. Part V: The eigenvalue structure of order one quasiseparable matrices; Part VI: Divide and conquer method for eigenproblem; Part VII: Algorithms for QR iterations and for reduction to Hessenberg form; Part VIII: QR iterations for companion matrices.

**To whom this book is addressed.** The book belongs to the area of theoretical and computational Linear Algebra. It is a self-contained monograph which has the structure of a graduate text. The main material was developed the last 30–40 years and is presented here following the lines and principles of a course in Linear Algebra. The book is based mostly on the relatively recent results obtained by
the authors and their coauthors. All these features together with many significant applications and accessible style will make it widely useful for engineers, scientists, numerical analysts, computer scientists and mathematicians alike.

**Acknowledgment.** We would like to express our gratitude the late Israel Koltracht and also Harry Dym, Rien Kaashoek, Thomas Kailath and Peter Lancaster with whom the work on semiseparable matrices has been started. It is also a pleasure to thank our colleagues Tom Bella, Dario Bini, Paola Boito, Patrick Dewilde, Luca Gemignani, Vadim Olshevsky, Victor Pan, Eugene Tyrtyshnikov, Marc Van Barel, Raf Vandebril, Hugo Woerdeman, Jianlin Xia and Pavel Zhlobich for fruitful discussions and cooperation. The authors acknowledge the help and understanding of the School of Mathematical Sciences at Tel-Aviv University and of the Nathan and Lilly Silver Family Foundation. We thank also the Israel Science Foundation for partial support of our work by a grant in the period from 1997 till 2000.