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Volume 38

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The Theory of the Moiré Phenomenon

Volume I: Periodic Layers

Second Edition

by

Isaac Amidror

Peripheral Systems Laboratory,
Ecole Polytechnique Fédérale de Lausanne (EPFL), Switzerland
To my parents
No one admires Fourier more than I do. It is the only entertaining mathematical work I ever saw. Its lucidity has always been admired. But it was more than lucid. It was luminous. Its light showed a crowd of followers the way to a heap of new physical problems.

Oliver Heaviside [Heaviside71 p. 32]

**Front cover image:** A heart-shaped moiré which is generated in the off-centered superposition of two circular gratings with slightly different radial periods. See Problem 11-8 and Fig. 11.4(c).

**Back cover images:** Interesting moiré effects in the superposition of two bell-shaped curvilinear gratings. See Figs. 10.34(c),(d).
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Preface to the Second Edition

Since the first edition of this book was published several new developments have been made in the field of the moiré theory. The most important of these concern new results that have recently been obtained on moiré effects between correlated aperiodic (or random) structures, a subject that was completely absent in the first edition, and which appears now for the first time in a second, separate volume.

This also explains the change in the title of the present volume, which now includes the subtitle “Volume I: Periodic Layers”. This subtitle has been added to clearly distinguish the present volume from its new companion, which is subtitled “Volume II: Aperiodic Layers”. It should be noted, however, that the new subtitle of the present volume may be somewhat misleading, since this book also treats (in Chapters 10 and 11) moiré effects between repetitive layers, which are, in fact, geometric transformations of periodic layers, that are generally no longer periodic in themselves. The most suitable subtitle for the present volume would therefore have been “Periodic or Repetitive Layers”, but in the end we have decided on the shorter version.

Although this revised edition maintains the general structure of the original book, it also includes some important improvements. It provides additional topics that were not explicitly treated in the first edition, such as the hybrid (1,-1)-moire effects with 2D intensity profiles (now in Sec. C.14 of Appendix C), the moiré effects between hexagonal screens (now in Sec. C.15 of Appendix C) or the extension of the indicial equations method to the case of 2D screens (in Sec. 11.2). The present edition of the book also includes several new figures and some new or revised problems. New references have been added throughout the book, and all the Internet references have been verified and updated. And finally, cross-references have been added wherever appropriate to the second volume, and in particular to those of its appendices which may be of interest to readers of the present book. Note, however, that the two volumes are basically independent of each other. Each volume thus contains its own Glossary, List of notations and symbols, References and Index.

In preparing this second edition, we have also taken the opportunity to correct errors and typos that crept into the original edition of the book. However, some errors may have been
overlooked, and some may have been inadvertently added in this new edition. Such errors, when detected, will be listed along with their corrections in the Internet site of the book, and we therefore encourage readers to inform us of any errors they may find.


This work would not have been possible without the support and the excellent research environment provided by the EPFL. In particular, the author wishes to express his gratitude to Prof. Roger D. Hersch, the head of the Peripheral Systems Laboratory of the EPFL, for his encouragement throughout the different stages of this project. Many thanks are also due to the publishers for their helpfulness and availability throughout the publishing cycle.
From the Preface to the First Edition

Who has not noticed, on one occasion or another, those intriguing geometric patterns which appear at the intersection of repetitive structures such as two far picket fences on a hill, the railings on both sides of a bridge, superposed layers of fabric, or folds of a nylon curtain? This fascinating phenomenon, known as the moiré effect, has found useful applications in several fields of science and technology, such as metrology, strain analysis or even document authentication and anti-counterfeiting. However, in other situations moiré patterns may have an unwanted, adverse effect. This is the case in the printing world, and, in particular, in the field of colour reproduction: moiré patterns which may be caused by the dot-screens used for colour printing may severely deteriorate the image quality and turn into a real printer’s nightmare.

The starting point of the work on which this book is based was, indeed, in the research of moiré phenomena in the context of the colour printing process. The initial aim of this research was to understand the nature and the causes of the superposition moiré patterns between regular screens in order to find how to avoid, or at least minimize, their adverse effect on colour printing. This interesting research led us, after all, to a much more far-reaching mathematical understanding of the moiré phenomenon, whose interest stands in its own right, independently of any particular application. Based on these results, the present book offers a profound insight into the moiré phenomenon and a solid theoretical basis for its full understanding. In addition to the question of moiré minimization between regular screens, the book covers many interesting and important subjects such as the navigation in the moiré parameter space, the intensity profile forms of the moiré, its singular states, its periodic or almost-periodic properties, the phase of the superposed layers and of each of the eventual moirés, the relations between macro- and micro-structures in the superposition, polychromatic moirés between colour layers, etc. All this is done in the most general way for any number of superposed layers having any desired forms (line-gratings, dot-screens with any dot shape, etc.). The main aim of this book is, therefore, to present all this material in the form of a single, unified and coherent text, starting from the basics of the theory, but also going in depth into recent research results and showing the new insight they offer in the understanding of the moiré phenomenon.
Fourier-based tools are but a natural choice when dealing with periodic phenomena; and, indeed, our approach is largely based on the Fourier theory. We consider each of the superposed layers as a function (reflectance or transmittance function) having values in the range between 0 and 1. We study the original layers, their superpositions, and their moiré effects by analyzing their properties both in the image domain and in the spectral, frequency domain using the Fourier theory. Further results are obtained by investigating the spectrum using concepts from geometry of numbers and linear algebra, and by interpreting the corresponding image-domain properties by means of the theories of periodic and almost-periodic functions. However, no prior knowledge of these fields of mathematics is assumed, and the required background is fully introduced in the text (in Chapter 5 and in Appendices A and B, respectively). The only prerequisite mathematical background is limited to undergraduate mathematics and an elementary familiarity with the Fourier theory (Fourier series, Fourier transforms, convolutions, Dirac impulses, etc.).

This book presents a comprehensive approach that provides a full explanation of the various phenomena which occur in the superposition, both in the image and in the spectral domains. This includes not only a quantitative and qualitative analysis of the moiré effect, but also the synthesis of moiré effects having any desired geometric forms and intensity profiles. In the first chapters we present the basic theory which covers the most fundamental case, namely: the superposition of monochrome, periodic layers. In later chapters of the book we extend the theory to the even more fascinating cases of polychromatic moirés and moirés between repetitive, non-periodic layers. Throughout the whole text we favour a pictorial, intuitive approach supported by mathematics, and the discussion is accompanied by a large number of figures and illustrative examples, some of which are visually striking and even spectacular.

This book is intended for students, scientists, and engineers wishing to widen their knowledge of the moiré effect; on the other hand it also offers a beautiful demonstration of the Fourier theory and its relationship with other fields of mathematics and science. Teachers and students of imaging science will find moiré phenomena to be an excellent didactic tool for illustrating the Fourier theory and its practical applications in one or more dimensions (Fourier transforms, Fourier series, convolutions, etc.). People interested in the various moiré applications and moiré-based technologies will find in this book a theoretical explanation of the moiré phenomenon and its properties. Readers interested in mathematics will find in the book a novel approach combining Fourier theory and geometry of numbers; physicists and crystallographers may be interested in the intricate relationship between the macro- and microstructures in the superposition and their relation to the theories of periodic and almost-periodic functions; and colour scientists and students will find in the polychromatic moirés a vivid demonstration of the additive and subtractive principles of colour theory. Finally, the occasional reader will enjoy the beauty of the effects demonstrated throughout this book, and it is our hope may be tempted to learn more about their nature and their properties.