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To Cheng, Sunny, Andrew, and Alan
Preface to the Second Edition

The first edition of this book appeared a decade ago. This is a revised expanded version. My goal has remained the same: to provide a text for a second course in matrix theory and linear algebra accessible to advanced undergraduate and beginning graduate students. Through the course, students learn, practice, and master basic matrix results and techniques (or matrix kung fu) that are useful for applications in various fields such as mathematics, statistics, physics, computer science, and engineering, etc.

Major changes for the new edition are: eliminated errors, typos, and mistakes found in the first edition; expanded with topics such as matrix functions, nonnegative matrices, and (unitarily invariant) matrix norms; included more than 1000 exercise problems; rearranged some material from the previous version to form a new chapter, Chapter 4, which now contains numerical ranges and radii, matrix norms, and special operations such as the Kronecker and Hadamard products and compound matrices; and added a new chapter, Chapter 10, “Majorization and Matrix Inequalities”, which presents a variety of inequalities on the eigenvalues and singular values of matrices and unitarily invariant norms.

I am thankful to many mathematicians who have sent me their comments on the first edition of the book or reviewed the manuscript of this edition: Liangjun Bai, Jane Day, Farid O. Farid, Takayuki Furuta, Geoffrey Goodson, Roger Horn, Zejun Huang, Minghua Lin, Dennis Merino, George P. H. Styan, Götz Trenkler, Qingwen Wang, Yimin Wei, Changqing Xu, Hu Yang, Xingzhi Zhan, Xiaodong Zhang, and Xiuping Zhang. I also thank Farquhar College of Arts and Sciences at Nova Southeastern University for providing released time for me to work on this project.

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Readers are welcome to communicate with me via e-mail.

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Preface

It has been my goal to write a concise book that contains fundamental ideas, results, and techniques in linear algebra and (mainly) in matrix theory which are accessible to general readers with an elementary linear algebra background. I hope this book serves the purpose.

Having been studied for more than a century, linear algebra is of central importance to all fields of mathematics. Matrix theory is widely used in a variety of areas including applied math, computer science, economics, engineering, operations research, statistics, and others.

Modern work in matrix theory is not confined to either linear or algebraic techniques. The subject has a great deal of interaction with combinatorics, group theory, graph theory, operator theory, and other mathematical disciplines. Matrix theory is still one of the richest branches of mathematics; some intriguing problems in the field were long standing, such as the Van der Waerden conjecture (1926–1980), and some, such as the permanental-dominance conjecture (since 1966), are still open.

This book contains eight chapters covering various topics from similarity and special types of matrices to Schur complements and matrix normality. Each chapter focuses on the results, techniques, and methods that are beautiful, interesting, and representative, followed by carefully selected problems. Many theorems are given different proofs. The material is treated primarily by matrix approaches and reflects the author’s tastes.

The book can be used as a text or a supplement for a linear algebra or matrix theory class or seminar. A one-semester course may consist of the first four chapters plus any other chapter(s) or section(s). The only prerequisites are a decent background in elementary linear algebra and calculus (continuity, derivative, and compactness in a few places). The book can also serve as a reference for researchers and instructors.

The author has benefited from numerous books and journals, including The American Mathematical Monthly, Linear Algebra and Its Applications, Linear and Multilinear Algebra, and the International Linear Algebra Society (ILAS) Bulletin Image. This book would not exist without the earlier works of a great number of authors (see the References).

I am grateful to the following professors for many valuable suggestions and input and for carefully reading the manuscript so that many errors have been eliminated from the earlier version of the book:

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## Contents

Preface to the Second Edition ........................................... vii  
Preface ................................................................. ix  
Frequently Used Notation and Terminology ............................... xv  
Frequently Used Theorems ............................................. xvii  

1 Elementary Linear Algebra Review ........................................ 1  
1.1 Vector Spaces ..................................................... 1  
1.2 Matrices and Determinants .......................................... 8  
1.3 Linear Transformations and Eigenvalues .............................. 17  
1.4 Inner Product Spaces ............................................. 27  

2 Partitioned Matrices, Rank, and Eigenvalues ............................ 35  
2.1 Elementary Operations of Partitioned Matrices ....................... 35  
2.2 The Determinant and Inverse of Partitioned Matrices ............... 42  
2.3 The Rank of Product and Sum ..................................... 51  
2.4 The Eigenvalues of $AB$ and $BA$ ................................ 57  
2.5 The Continuity Argument and Matrix Functions .................... 62  
2.6 Localization of Eigenvalues: The Geršgorin Theorem ................ 67  

3 Matrix Polynomials and Canonical Forms ................................ 73  
3.1 Commuting Matrices .............................................. 73  
3.2 Matrix Decompositions ............................................ 79  
3.3 Annihilating Polynomials of Matrices ................................ 87  
3.4 Jordan Canonical Forms .......................................... 93  
3.5 The Matrices $A^T$, $A^*$, $A^TA$, $A^*A$, and $AA$ ............... 102  

4 Numerical Ranges, Matrix Norms, and Special Operations ............ 107  
4.1 Numerical Range and Radius ...................................... 107  
4.2 Matrix Norms ..................................................... 113  
4.3 The Kronecker and Hadamard Products ............................ 117  
4.4 Compound Matrices .............................................. 122
5 Special Types of Matrices 125
5.1 Idempotence, Nilpotence, Involution, and Projections ...... 125
5.2 Tridiagonal Matrices .................................. 133
5.3 Circulant Matrices .................................. 138
5.4 Vandermonde Matrices .................................. 143
5.5 Hadamard Matrices .................................. 150
5.6 Permutation and Doubly Stochastic Matrices .......... 155
5.7 Nonnegative Matrices .................................. 164

6 Unitary Matrices and Contractions 171
6.1 Properties of Unitary Matrices .......................... 171
6.2 Real Orthogonal Matrices .................................. 177
6.3 Metric Space and Contractions .......................... 182
6.4 Contractions and Unitary Matrices ..................... 188
6.5 The Unitary Similarity of Real Matrices ............. 192
6.6 A Trace Inequality of Unitary Matrices ........... 195

7 Positive Semidefinite Matrices 199
7.1 Positive Semidefinite Matrices .......................... 199
7.2 A Pair of Positive Semidefinite Matrices .............. 207
7.3 Partitioned Positive Semidefinite Matrices ........... 217
7.4 Schur Complements and Determinant Inequalities ... 227
7.5 The Kronecker and Hadamard Products
   of Positive Semidefinite Matrices ..................... 234
7.6 Schur Complements and the Hadamard Product ...... 240
7.7 The Wielandt and Kantorovich Inequalities ........ 245

8 Hermitian Matrices 253
8.1 Hermitian Matrices and Their Inertias ............. 253
8.2 The Product of Hermitian Matrices .................. 260
8.3 The Min-Max Theorem and Interlacing Theorem ...... 266
8.4 Eigenvalue and Singular Value Inequalities .......... 274
8.5 Eigenvalues of Hermitian matrices $A$, $B$, and $A + B$ ...... 281
8.6 A Triangle Inequality for the Matrix $(A^*A)^{1/2}$  ........ 287

9 Normal Matrices 293
9.1 Equivalent Conditions .................................. 293
9.2 Normal Matrices with Zero and One Entries .......... 306
9.3 Normality and Cauchy–Schwarz–Type Inequality ...... 312
9.4 Normal Matrix Perturbation .......................... 319
## 10 Majorization and Matrix Inequalities

<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>10.1</td>
<td>Basic Properties of Majorization</td>
<td>325</td>
</tr>
<tr>
<td>10.2</td>
<td>Majorization and Stochastic Matrices</td>
<td>334</td>
</tr>
<tr>
<td>10.3</td>
<td>Majorization and Convex Functions</td>
<td>340</td>
</tr>
<tr>
<td>10.4</td>
<td>Majorization of Diagonal Entries, Eigenvalues, and Singular Values</td>
<td>349</td>
</tr>
<tr>
<td>10.5</td>
<td>Majorization for Matrix Sum</td>
<td>356</td>
</tr>
<tr>
<td>10.6</td>
<td>Majorization for Matrix Product</td>
<td>363</td>
</tr>
<tr>
<td>10.7</td>
<td>Majorization and Unitarily Invariant Norms</td>
<td>372</td>
</tr>
</tbody>
</table>

### References

- References ................................................................. 379
- Notation ................................................................. 391
- Index .............................................................................. 395
Frequently Used Notation and Terminology

\text{dim} V, 3 \quad \text{dimension of vector space } V

\mathbb{M}_n, 8 \quad n \times n \text{ (i.e., } n\text{-square) matrices with complex entries}

A = (a_{ij}), 8 \quad \text{matrix } A \text{ with } (i, j)\text{-entry } a_{ij}

I, 9 \quad \text{identity matrix}

A^T, 9 \quad \text{transpose of matrix } A

A, 9 \quad \text{conjugate of matrix } A

A^*, 9 \quad \text{conjugate transpose of matrix } A, \text{ i.e., } A^* = \overline{A}^T

A^{-1}, 13 \quad \text{inverse of matrix } A

\text{rank } (A), 11 \quad \text{rank of matrix } A

\text{tr } A, 21 \quad \text{trace of matrix } A

\det A, 12 \quad \text{determinant of matrix } A

\left| A \right|, 12, 83, 164 \quad \text{determinant for a block matrix } A \text{ or } A^*A^{1/2} \text{ or } (|a_{ij}|)

(u, v), 27 \quad \text{inner product of vectors } u \text{ and } v

\| \cdot \|, 28, 113 \quad \text{norm of a vector or a matrix}

\text{Ker}(A), 17 \quad \text{kernel or null space of } A, \text{ i.e., } \text{Ker}(A) = \{x : Ax = 0\}

\text{Im}(A), 17 \quad \text{image space of } A, \text{ i.e., } \text{Im}(A) = \{Ax\}

\rho(A), 109 \quad \text{spectral radius of matrix } A

\sigma_{\text{max}}(A), 109 \quad \text{largest singular value (spectral norm) of matrix } A

\lambda_{\text{max}}(A), 124 \quad \text{largest eigenvalue of matrix } A

A \geq 0, 81 \quad A \text{ is positive semidefinite} \quad \text{(or all } a_{ij} \geq 0 \text{ in Section 5.7)}

A \geq B, 81 \quad A - B \text{ is positive semidefinite} \quad \text{(or } a_{ij} \geq b_{ij} \text{ in Section 5.7)}

A \circ B, 117 \quad \text{Hadamard (entrywise) product of matrices } A \text{ and } B

A \otimes B, 117 \quad \text{Kronecker (tensor) product of matrices } A \text{ and } B

x \prec_w y, 326 \quad \text{weak majorization}, \text{ i.e., all } \sum_{i=1}^{k} x_i^k \leq \sum_{i=1}^{k} y_i^k \text{ hold}

x \prec_{\text{wlog}} y, 344 \quad \text{weak log-majorization}, \text{ i.e., all } \prod_{i=1}^{k} x_i^k \leq \prod_{i=1}^{k} y_i^k \text{ hold}

An \, n \times n \, \text{matrix } A \text{ is said to be}

upper-triangular \quad \text{if all entries below the main diagonal are zero}

diagonalizable \quad \text{if } P^{-1}AP \text{ is diagonal for some invertible matrix } P

similar to } B \quad \text{if } P^{-1}AP = B \text{ for some invertible matrix } P

unitarily similar to } B \quad \text{if } U^*AU = B \text{ for some unitary matrix } U

unitary \quad \text{if } AA^* = A^*A = I, \text{ i.e., } A^{-1} = A^*

positive semidefinite \quad \text{if } x^*Ax \geq 0 \text{ for all vectors } x \in \mathbb{C}^n

Hermitian \quad \text{if } A = A^*

normal \quad \text{if } A^*A = AA^*

\lambda \in \mathbb{C} \text{ is an eigenvalue of } A \in \mathbb{M}_n \text{ if } Ax = \lambda x \text{ for some nonzero } x \in \mathbb{C}^n.
• **Cauchy–Schwarz inequality:** Let $V$ be an inner product space over a number field ($\mathbb{R}$ or $\mathbb{C}$). Then for all vectors $x$ and $y$ in $V$

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle.$$ 

Equality holds if and only if $x$ and $y$ are linearly dependent.

• **Theorem on the eigenvalues of $AB$ and $BA$:** Let $A$ and $B$ be $m \times n$ and $n \times m$ complex matrices, respectively. Then $AB$ and $BA$ have the same nonzero eigenvalues, counting multiplicity. As a consequence,

$$\text{tr}(AB) = \text{tr}(BA).$$

• **Schur triangularization theorem:** For any $n$-square matrix $A$, there exists an $n$-square unitary matrix $U$ such that $U^*AU$ is upper-triangular.

• **Jordan decomposition theorem:** For any $n$-square matrix $A$, there exists an $n$-square invertible complex matrix $P$ such that

$$A = P^{-1}(J_1 \oplus J_2 \oplus \cdots \oplus J_k)P,$$

where each $J_i$, $i = 1, 2, \ldots, k$, is a Jordan block.

• **Spectral decomposition theorem:** Let $A$ be an $n$-square normal matrix with eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Then there exists an $n$-square unitary matrix $U$ such that

$$A = U^* \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)U.$$ 

In particular, if $A$ is positive semidefinite, then all $\lambda_i \geq 0$; if $A$ is Hermitian, then all $\lambda_i$ are real; and if $A$ is unitary, then all $|\lambda_i| = 1$.

• **Singular value decomposition theorem:** Let $A$ be an $m \times n$ complex matrix with rank $r$. Then there exist an $m$-square unitary matrix $U$ and an $n$-square unitary matrix $V$ such that

$$A = UDV,$$

where $D$ is the $m \times n$ matrix with $(i, i)$-entries being the singular values of $A$, $i = 1, 2, \ldots, r$, and other entries 0. If $m = n$, then $D$ is diagonal.