PART A

\((m, s)\)-SPLINES
INTRODUCTION

The theory of \((m,s)\)-splines was introduced and developed by J. Duchon [53, 54, 55, 56, 57]. The study is situated in the framework of certain functional spaces of the “Sobolev type”, the spaces \(X^{m,s}\). These spaces are semi-Hilbert spaces, i.e. vector spaces endowed with a scalar semi-product and the associated semi-norm, and which are complete for this semi-norm.

The method elaborated by J. Duchon follows M. Attéia’s [22, 23] ideas about abstract spline functions: it appeals to the notion of reproducing kernel of a semi-Hilbert space (cf. N. Aronszajn [21], L. Schwartz [130]). This method provides an explicit characterization of the \((m,s)\)-splines from the knowledge of a reproducing kernel of the space \(X^{m,s}\). A part of the results can be deduced directly from P.-J. Laurent [88, Chapter 4].

The point of view adopted in our work is different: we do not use reproducing kernels, but we treat the spaces \(X^{m,s}\) as Hilbert spaces, by equipping them with suitable norms. This method allows us to obtain all the results in a relatively simple way and, on the other hand, it appears well adapted to establish error estimates.

Chapter I is devoted to the study of properties of the spaces \(X^{m,s}\). In particular, we define (following [109]) a norm which makes \(X^{m,s}\) a Hilbert space. We point out that the spaces introduced in this part are spaces of complex valued functions, due to the use of the Fourier transform.

Chapter II discusses interpolating splines, essentially for the model problem of Lagrange interpolation. By endowing \(X^{m,s}\) with a norm associated with the interpolation conditions, we establish with no difficulty the existence, the uniqueness and two characterizations of the interpolating \((m,s)\)-splines. To get them explicitly, we make use of the Fourier transform of functions of the Euclidean distance: the corresponding result is obtained by solving a problem of division in \(S'\). We then give some examples: thin plate splines, pseudo-polynomial splines, and splines defined by local mean values. We obtain the linear system which determines the interpolating spline and we verify that the matrix of the system is regular. Using a convenient norm for the space \(X^{m,s}\), we show the convergence of the interpolating \((m,s)\)-splines. Finally, by means of several technical results, we can establish estimates of the interpolation error for a function belonging to the Sobolev space \(H^{m+s}(\Omega)\) in terms of the Hausdorff distance \(d\) between \(\Omega\) and the set of data points.

Chapter III reconsiders for smoothing splines the questions which are treated in Chapter II for the interpolating splines: existence, uniqueness, characterization and computation. Likewise, we prove the convergence of the smoothing splines to an interpolating spline when the smoothing parameter \(\varepsilon\) tends to 0. We then prove the convergence of the smoothing splines and we establish error estimates. Finally, we study the problem of convergence for noisy data.

Chapter IV gives a résumé of the \((m,l,s)\)-splines, which are a generalization of the \((m,s)\)-splines.