Bilinear Integrable Systems: From Classical to Quantum, Continuous to Discrete
NATO Science Series

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Series II: Mathematics, Physics and Chemistry – Vol. 201
Bilinear Integrable Systems: From Classical to Quantum, Continuous to Discrete

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PREFACE

On April 29, 1814 Napoleon landed on the island of Elba, surrounded with a personal army of 1200 men. The allies, Russia, Prussia, England and Austria, had forced him into exile after a number of very costly defeats; he was deprived of all his titles, but could keep the title of “Emperor of Elba”. History tells us that each morning he took long walks in the sun, reviewed his army each midday and discussed world matters with newly appointed advisors, following the same pattern everyday, to the great surprise of Campbell, the British officer who was to keep an eye on him. All this made everyone believe he was settled there for good. Napoleon once said: Elba is beautiful, but a bit small. Elba was definitely a source of inspiration; indeed, the early morning, March 6, 1815, Metternich, the chancellor of Austria was woken up by one of his aides with the stunning news that Napoleon had left Elba with his 1200 men and was marching to Paris with little resistance; A few days later he took up his throne again in the Tuileries. In spite of his insatiable hunger for battles and expansion, he is remembered as an important statesman. He was a pioneer in setting up much of the legal, administrative and political machinery in large parts of continental Europe.

We gathered here in a lovely and quaint fishing port, Marciana Marina on the island of Elba, to celebrate one of the pioneers of integrable systems, Hirota Sensei, and this at the occasion of his seventieth birthday. Trained as a physicist in his home university Kyushu University, Professor Hirota earned his PhD in ’61 at Northwestern University with Professor Siegert in the field of “Quantum Statistical mechanics”. He wrote a widely appreciated Doctoral dissertation on “Functional Integral representation of the grand partition function”. As a young researcher, he entered the RCA Company in Tokyo to do research on semiconductor plasmas. He then joined the Faculty of Science and Engineering of Ritsumeikan University in Kyoto and then later Hiroshima National University and Waseda University, until his recent retirement.

We are also celebrating another birthday, namely the birth, some thirty years ago, of multisoliton solutions for the KdV equation, the representation of integrable equations as bilinear equation and Hirota’s D-operation. All this happened in the period 1971 through 1974.

Professor Hirota was led to model the Toda lattice as a non-linear network of ladder-type LC circuits. The self-dual case led to equations very reminiscent of the Sine-Gordon equation, with much the same features (existence of one soliton, soliton-soliton interaction, etc)
Meanwhile, At RCA, Hirota Sensei was looking for applications of solitons to multi-channel communication systems. As an important requirement, they needed to be stable in the presence of a ripple. Taking a 2-soliton interaction, letting one of them become very small, led to the stability of a 1-soliton solutions. What about the stability of two solitons? Professor Hirota argued as follows: If one wants to use the same method, one should look for three-soliton solutions and again let one soliton become very small. In the beginning, most naïve guesses turned out to be wrong. Finally the answer came from an ingenious use of the Bäcklund transformation and a superposition principle, for the sine-Gordon equation. In this way, Professor Hirota expressed the three-soliton solution, in terms of sums of exponentials with phases linear in $x$ and $t$. These same kind of methods could then be applied to the non-linear self-dual network equation, the Toda equation and finally to the KdV equation.

In his celebrated 1971-paper: “Exact solutions of the KdV equation for multiple collisions of solitons”, Hirota gave the multisoliton solution to the KdV equation in terms of the second logarithmic derivative of a determinant of exponentials and showed most importantly that the determinant satisfies a bilinear equation of order 4. So Hirota’s bilinear equation was born.

The story goes that Professor Scott who was visiting Japan in the summer 1971 remarked: why do you want to replace the KdV equation by a much more complicated equation, namely the bilinear equation, which after all is 4th order? This seemingly negative comment had striking consequences. Having written bilinear equations for all those integrable PDE’s, Professor Hirota became very concerned with finding simple ways to express them, which he did in a paper in 1974, where he introduced the operation, known these days as Hirota symbol or Hirota D-operator. This amazing intuition turned out to have profound consequences. Beyond being an ingenious device, it had a lasting impact onto the field. It gave rise to the famous tau-function theory, which by now has become a classic chapter of mathematical physics. One might say that the Hirota symbol has become one of those tools that everyone is using without referring to it in the bibliography, just like Schwarz’s inequality or Stokes’ theorem.

Hirota’s career is specked with striking and stunning discoveries, often based on simple, but ingenious observations. They unleashed a great tide of energy and activity; all hell broke loose. In the 70’s, one miracle came after the other, the field literally exploded in the most fascinating directions that we all know and worship. This week here in Elba will be a tribute to his work!

This NATO-sponsored workshop here in Elba was dominated by an enormous wealth of subjects around integrability, ranging from geometric to analytic questions, from Lie groups, quantum groups and W-algebras to combinatorics and quantum field theory. We would like to thank the participants for having delivered these interesting lectures. Also many thanks to those who have contributed to this volume.
The organizing committee consisted of Professors Franklin Lambert, Frank Nijhoff, Ludwig Faddeev and Pierre van Moerbeke. Last but not least, we would like to express our gratitude to Professor Franklin Lambert. It was his idea to organize the conference on this theme, he picked this wonderful spot, he was the real engine behind this enterprise, he did an enormous amount of work. Thank you Franklin!

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