Part III

KASHIWARA’S OPERATORS
AND APPLICATIONS

In the author’s elementary algebraic definition [4] of the canonical basis of \( f \), there were three main ingredients: (a) the basis was assumed to be integral in a suitable sense; (b) the basis was assumed fixed by the involution \( \cdot \); (c) the basis was assumed to be in a specified \( \mathbb{Z}[v^{-1}] \)-lattice \( \mathcal{L} \) and had prescribed image in \( \mathcal{L}/v^{-1}\mathcal{L} \).

Of these three ingredients, the last one is the most subtle; in [4], \( \mathcal{L} \) and the basis of \( \mathcal{L}/v^{-1}\mathcal{L} \) were defined in terms of a braid group action. This definition does not work for Cartan data of infinite type.

Kashiwara’s scheme [2] to define a basis of \( f \) involves again the ingredients (a),(b),(c) above, but he proposes a quite different way to construct the lattice \( \mathcal{L} \) and the basis of \( \mathcal{L}/v^{-1}\mathcal{L} \), which makes sense for any Cartan datum. The main ingredients in his definition were certain operators \( \check{\epsilon}_i, \check{\phi}_i : f \to f \) and some analogous operators \( \check{E}_i, \check{F}_i \) on any integrable \( U \)-module. (The last operators were already introduced, in a dual form, in an earlier paper [1].)

Part III gives an account of Kashiwara’s approach and its applications. (The results in Part III will be needed in Part IV.) Our exposition differs from that of Kashiwara to some extent. In particular, we will make use of the existence of canonical bases (up to sign) established in Part II, while for Kashiwara, that existence was one of the goals.

The algebra \( U \) in Chapter 15 is defined in a different way than in [2], but eventually, the two definitions agree. The operators \( \check{\epsilon}_i, \check{\phi}_i, \check{E}_i, \check{F}_i \) are defined in Chapter 16. Chapter 17 contains a proof of a crucial result of Kashiwara on the behaviour of \( \check{E}_i, \check{F}_i \) in a tensor product. Chapters 18 and 19 are concerned with various properties of the canonical basis of \( \Lambda_\lambda \), in particular with the fact that this basis is almost orthonormal for the natural inner product. Chapter 20 deals with bases at \( \infty \) (or crystal bases in Kashiwara’s terminology). Chapter 21 deals with the special features which hold in the case where the Cartan datum is of finite type. Chapter 22 contains some new positivity results.

In the remainder of this book we assume that, unless otherwise specified, a Cartan datum \((I, \cdot)\) and a root datum \((Y, X, \ldots)\) of type \((I, \cdot)\) have been fixed. The notation \( f, U, \) etc. will refer to these fixed data.