Topics in Hyperplane Arrangements, Polytopes and Box-Splines
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The main purpose of this book is to bring together some areas of research that have developed independently over the last 30 years. The central problem we are going to discuss is that of the computation of the number of integral points in suitable families of variable polytopes. This problem is formulated in terms of the study of partition functions. The partition function $T_X(b)$, associated to a finite set of integral vectors $X$, counts the number of ways in which a variable vector $b$ can be written as a linear combination of the elements in $X$ with positive integer coefficients. Since we want this number to be finite, we assume that the vectors $X$ generate a pointed cone $C(X)$.

Special cases were studied in ancient times, and one can look at the book of Dickson [50] for historical information on this topic.

The problem goes back to Euler in the special case in which $X$ is a list of positive integers, and in this form it was classically treated by several authors, such as Cayley, Sylvester [107] (who calls the partition function the *quotity*), Bell [15] and Ehrhart [53], [54].

Having in mind only the principal goal of studying the partition functions, we treat several topics but not in a systematic way, by trying to show and compare a variety of different approaches. In particular, we want to revisit a sequence of papers of Dahmen and Micchelli, which for our purposes, culminate in the proof of a slightly weaker form of Theorem 13.54, showing the quasipolynomial nature of partition functions [37] on suitable regions of space.

The full statement of Theorem 13.54 follows from further work of Szenes and Vergne [110].

This theory was approached in a completely different way a few years later by various authors, unaware of the work of Dahmen and Micchelli. We present an approach taken from a joint paper with M. Vergne [44] in Section 13.4.5, which has proved to be useful for further applications to the index theory.

In order to describe the regions where the partition function is a quasipolynomial, one needs to introduce a basic geometric and combinatorial object: the zonotope $B(X)$ generated by $X$. This is a compact polytope defined in
2.12. The theory then consists in dividing \( C(X) \) into regions \( \Omega \), called big cells, such that in each region \( \Omega - B(X) \), the partition function is a quasipolynomial (see Definition 5.31).

The quasipolynomials appearing in the description of the partition function satisfy a natural system of difference equations of a class that in the multidimensional case we call Eulerian (generalizing the classical one-dimensional definition); see Theorem 5.32. All these results can be viewed as generalizations of the theory of the Ehrhart polynomial [53], [54].

The approach of Dahmen and Micchelli to partition functions is inspired by their study of two special classes of functions: the multivariate spline \( T_X(x) \), supported on \( C(X) \), and the box spline \( B_X(x) \), supported on \( B(X) \), originally introduced by de Boor and deVore [39]; see Section 7.1.1 for their definition.

These functions, associated to the given set of vectors \( X \), play an important role in approximation theory. One of the goals of the theory is to give computable closed formulas for all these functions and at the same time to describe some of their qualitative behavior and applications.

These three functions can be described in a combinatorial way as a finite sum over local pieces (see formulas (9.5) and (14.28)). In the case of \( B_X(x) \) and \( T_X(x) \) the local pieces span, together with their derivatives, a finite-dimensional space \( D(X) \) of polynomials. In the case of \( T_X(b) \) they span, together with their translates, a finite-dimensional space \( DM(X) \) of quasipolynomials.

A key fact is the description of:

- \( D(X) \) as solutions of a system of differential equations by formula (11.1).
- \( DM(X) \) as solutions of a system of difference equations by formula (13.3).
- A strict relationship between \( D(X) \) and \( DM(X) \) in Section 16.1.

In particular, Dahmen and Micchelli compute the dimensions of both spaces, see Theorem 11.8 and 13.21. This dimension has a simple combinatorial interpretation in terms of \( X \). They also decompose \( DM(X) \) as a direct sum of natural spaces associated to certain special points \( P(X) \) in the torus whose character group is the lattice spanned by \( X \). In this way, \( DM(X) \) can be identified with a space of distributions supported at these points. Then \( D(X) \) is the subspace of \( DM(X) \) of the elements supported at the identity. The papers of Dahmen and Micchelli are a development of the theory of splines, initiated by I.J. Schoenberg [95]. There is a rather large literature on these topics by several authors, such as A.A. Akopyan; A. Ben-Artzi, C.K. Chui, C. De Boor, H. Diamond, N. Dyn, K. Höllig, Rong Qing Jia, A. Ron, and A.A. Saakyan. The interested reader can find a lot of useful historical information about these matters and further references in the book [40] (and also the notes of Ron [93]).

The results about the spaces \( D(X) \) and \( DM(X) \), which, as we have mentioned, originate in the theory of splines, turn out to have some interest also in
the theory of hyperplane arrangements and in commutative algebra in connection with the study of certain Reisner–Stanley algebras [43]. Furthermore, the space $DM(X)$ has an interpretation in the theory of the index of transversally elliptic operators (see [44]).

The fact that a relationship between this theory and hyperplane arrangements should exist is pretty clear once we consider the set of vectors $X$ as a set of linear equations that define an arrangement of hyperplanes in the space dual to that in which $X$ lies. In this respect we have been greatly inspired by the results of Orlik–Solomon on cohomology [84], [83] and those of Brion, Szenes, Vergne on partition functions [109], [110], [27], [22], [28], [29], [26], [108].

In fact, a lot of work in this direction originated from the seminal paper of Khovanskii and Pukhlikov [90] interpreting the counting formulas for partition functions as Riemann–Roch formulas for toric varieties, and of Jeffrey–Kirwan [68] and Witten [120], on moment maps. These topics are beyond the scope of this book, which tries to remain at a fairly elementary level. For these matters the reader may refer to Vergne’s survey article [116].

Due to the somewhat large distance between the two fields, people working in hyperplane arrangements do not seem to be fully aware of the results on the box spline.

On the other hand, there are some methods that have been developed for the study of arrangements that we believe shed some light on the space of functions used to build the box spline. Therefore, we feel that this presentation may be useful in making a bridge between the two theories.

For completeness and also to satisfy our personal curiosity, we have also added a short discussion on the applications of box splines to approximation theory and in particular the aspect of the finite element method, which comes from the Strang–Fix conditions (see [106]). In Section 18.1 we present a new approach to the construction of quasi-interpolants using in a systematic way the concept of superfunction.

Here is a rough description of the method we shall follow to compute the functions that are the object of study of this book.

- We interpret all the functions as tempered distributions supported in the pointed cone $C(X)$.
- We apply the Laplace transform and change the problem to one in algebra, essentially a problem of developing certain special rational functions into partial fractions.
- We solve the algebraic problems by module theory under the algebra of differential operators or of difference operators.
- We interpret the results by inverting the Laplace transform directly.

The book consists of five parts.
In the first part we collect basic material on convex sets, combinatorics and polytopes, the Laplace and Fourier transforms, and the language of modules over the Weyl algebra. We then recall some simple foundational facts on suitable systems of partial differential equations with constant coefficients. We develop a general approach to linear difference equations and give a method of reduction to the differentiable case (Section 5.3). We discuss in some detail the classical Tutte polynomial of a matroid and we take a detour to compute such a polynomial in the special case of root systems. The reader is advised to use these chapters mainly as a tool and reference to read the main body of the book. In particular, Chapter 6 is used only in the fourth part.

In the second part, on the differentiable case, we start by introducing and studying the splines. We next analyze the coordinate ring of the complement of a hyperplane arrangement using the theory of modules over the Weyl algebra. We apply this analysis to the computation of the multivariate splines. We next give a simple proof of the theorem of Dahmen–Micchelli on the dimension of $D(X)$ using elementary commutative algebra (Theorem 11.13), and discuss the similar theory of $E$-splines due to Amos Ron [92].

After this, we discuss the graded dimension of the space $D(X)$ (Theorem 11.13) in terms of the combinatorics of bases extracted from $X$. This is quite similar to the treatment of Dyn and Ron [51]. The answer happens to be directly related to the classical Tutte polynomial of a matroid introduced in the first part. We next give an algorithmic characterization in terms of differential equations of a natural basis of the top-degree part of $D(X)$ (Proposition 11.10), from which one obtains explicit local expressions for $T_X$ (Theorem 9.7). We complete the discussion by presenting a duality between $D(X)$ and a subspace of the space of polar parts relative to the hyperplane arrangement associated to $X$ (Theorem 11.20), a space which can also be interpreted as distributions supported on the regular points of the associated cone $C(X)$.

The third part, on the discrete case, contains various extensions of the results of the second part in the case in which the elements in $X$ lie in a lattice. This leads to the study of toric arrangements, which we treat by module theory in a way analogous to the treatment of arrangements. Following a joint paper with M. Vergne [44], we discuss another interesting space of functions on a lattice that unifies parts of the theory and provides a conceptual proof of the quasipolynomial nature of partition functions on the sets $\Omega - B(X)$.

We next explain the approach (due mainly to Szenes–Vergne) (see Theorem 10.11 and Formula (16.3)) of computing the functions under study via residues.

We give an explicit formula relating partition functions to multivariate splines (Theorem 16.12) which is a reinterpretation, with a different proof, of the formula of Brion–Vergne [27]. We use it to discuss classical computations including Dedekind sums and generalizations.

As an application of our methods, we have included two independent chapters, one in the second and one in the third part, in which we explain how to
compute the de Rham cohomology for the complement of a hyperplane or of a toric arrangement.

The fourth and fifth parts essentially contain complements. The fourth part is independent of the third. In it we present a short survey of the connections and applications to approximation theory: the role of the Strang–Fix conditions and explicit algorithms used to approximate functions by splines, such as, for example, can be found in the book *Box Splines* [40]. We also discuss briefly some other applications, such as the theory of stationary subdivision.

The fifth and final part is completely independent of the rest of the book. It requires some basic algebraic geometry, and it is included because it justifies in a geometric way the theory of Jeffrey–Kirwan residues which we have introduced as a purely computational tool, and the regularization of certain integrals. Here we give an overview of how the residues appear in the so-called *wonderful models* of subspace arrangements. These are particularly nice compactifications of complements of hyperplane arrangements that can be used to give a geometric interpretation of the theory of residues.

As the reader will notice, there is a class of examples which we investigate throughout this book. These are *root systems*. We decided not to give the foundations of root systems, since several excellent introductions are available in the literature, such as, for example, [20] or [67].

Finally, let us point out that there is an overlap between this material and other treatments, as for instance the recent book by Matthias Beck and Sinai Robins, in the Springer Undergraduate Texts in Mathematics series: *Computing, the Continuous Discretely. Integer-Point Enumeration in Polyhedra* [14].

The actual computation of partition functions or of the number of points in a given polytope is a very difficult question due to the high complexity of the problem. A significant contribution is found in the work of Barvinok [12], [10], [8], [9], [11] and in the papers of Baldoni, De Loera, Cochet, Vergne, and others (see [6], [34], [48]). We will not discuss these topics here. A useful survey paper is the one by Jésus A. De Loera [47].

**Acknowledgments**

We cannot finish this preface without expressing our sincere thanks to various colleagues and friends. First, Michèle Vergne, who through various lectures and discussions arouse our interest in these matters. We also thank Welleda Baldoni and Charles Cochet for help in computational matters, Carla Manni for pointing out certain aspects of quasi-interpolants, Amos Ron for sharing his notes, and Carl de Boor for useful suggestions on the literature. Finally,
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Although we have made an effort to give proper attributions, we may have missed some important contributions due to the rather large span of the existing literature, and we apologize if we have done so.

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Corrado de Concini
Claudio Procesi
Notations

When we introduce a new symbol or definition we will use the convenient form :=, which means that the term introduced at its left is defined by the expression at its right.

A typical example is $P := \{ x \in \mathbb{N} \mid 2 \text{ divides } x \}$, which stands for *P is by definition the set of all natural numbers x such that 2 divides x.*

The symbol $\pi : A \to B$ denotes a mapping named $\pi$ from the set $A$ to the set $B$.

Given two sets $A, B$ we set

$$A \setminus B := \{ a \in A \mid a \notin B \}.$$  

We use the standard notation

$$\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$$

for the natural numbers (including 0), the integers, and the rational, real, and complex numbers.

When $V$ is a vector space we denote by $V^*$ its dual. The canonical pairing between $\phi \in V^*$ and $v \in V$ will be denoted by $\langle \phi \mid v \rangle$ or sometimes by $\phi(v)$ when we want to stress $\phi$ as a function.

For a finite set $A$ we denote by $|A|$ its cardinality.

Given two points $a, b$ in a real vector space $V$ we have the closed segment $[a, b] := \{ ta + (1-t)b, \ 0 \leq t \leq 1 \}$ the open segment $(a, b) := \{ ta + (1-t)b, \ 0 < t < 1 \}$, and the two half-open segments $[a, b) := \{ ta + (1-t)b, \ 0 < t \leq 1 \}, (a, b] := \{ ta + (1-t)b, \ 0 \leq t < 1 \}$.

The closure of a set $C$ in a topological space is denoted by $\overline{C}$.

The interior of a set $C$ in a topological space is denoted by $\overset{\circ}{C}$.

*Sets and lists:* Given a set $A$, we denote by $\chi_A$ its characteristic function. The notation $A := \{ a_1, \ldots, a_k \}$ denotes a set called $A$ with elements the $a_i$ while the notation $A := (a_1, \ldots, a_k)$ denotes a list called $A$ with elements the $a_i$. The elements $a_i$ appear in order and may be repeated. A sublist of $A$ is a list $(a_{i_1}, \ldots, a_{i_r})$ with $1 \leq i_1 < \cdots < i_r \leq k$.  

WARNING: By abuse of notation, when $X$ is a list we shall also denote a sublist $Y$ by writing $Y \subset X$ and by $X \setminus Y$ the list obtained from $X$ by removing the sublist $Y$. 