Singular Integrals and Fourier Theory on Lipschitz Boundaries

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This book is a sincere dedication to
Professor Alan McIntosh

Preface

From the idea to the content, this book is basically Alan McIntosh's theory. In this book, we state systemically the theory of singular integrals and Fourier multipliers on Lipschitz graphs and surfaces which stems largely from the famous "Coifman-McIntosh-Meyer Theorem" since 1980s. The book elaborates the basic framework, essential thoughts, and main results of the theory. At the same time, this book also serves as a comprehensive reference on recent developments of this topic.

The subject of Fourier multipliers on Lipschitz surfaces has a profound background in harmonic analysis and partial differential equations. When we study boundary value problems of second-order elliptic operators, we need to deal with L^2-boundedness of the Cauchy-type integral operators on Lipschitz curves γ. Because the kernel of the Cauchy integral is nonlinear and non-smooth, there exists an essential difficulty on the study of the corresponding singular integral operators. In 1977, by using techniques of complex analysis, C. P. Calderón first proved that the singular Cauchy integral operator is bounded on $L^2(\gamma)$ under the assumption that the Lipschitz constant is sufficiently small. For the general cases, R. Coifman, A. McIntosh, and Y. Meyer applied the method of multilinear operators to get rid of the restriction on the Lipschitz constant and obtained the L^2-boundedness of the singular Cauchy integral operator on any Lipschitz curve γ. In considering the L^p-boundedness, $1 < p < \infty$, of a linear or non-linear operator, from the view of point of harmonic analysis, its L^2-boundedness would be the core. In fact, the L^p-boundedness of an operator may be deduced from its L^2-boundedness by using the interpolation theorem and the duality of L^p.

The corresponding problem on higher dimensional spaces is the L^p-boundedness of the singular Cauchy integral operators on Lipschitz surfaces Σ. The increase of the spatial dimension requires to apply a truly original approach. To introduce a Cauchy-type complex structure on the Euclidean spaces $\mathbb{R}^n$, the most efficient way is to embed $\mathbb{R}^n$ into the Hamilton quaternions or the Clifford algebra $\mathbb{R}_{(n)}$. The L^2-boundedness of the singular integral operators with holomorphic kernels on the Lipschitz surfaces was proved independently by Li et al. [1] and Gaudry et al. [2]. In this book, we adopt the method of Gaudry et al.

There exists a one to one correspondence between the convolution integrals T_{ϕ} and the Fourier multipliers M_b. In 1994, C. Li, A. McIntosh, and T. Qian established an explicit and one-to-one correspondence between the Clifford monogenic kernels ϕ and the complex holomorphic symbols b on Lipschitz graphs Σ (see [3]), and obtained the Cauchy-Dunford functional calculus of the Dirac operator on Σ. Such functional calculus has three equivalent forms: the Cauchy-Dunford integrals, the singular integrals with holomorphic kernels, and the bounded holomorphic Fourier multipliers. Since 1996, T. Qian began to consider the analogy on the high-dimensional spheres, tours, and their Lipschitz perturbations, i.e., the theories on starlike Lipschitz surfaces. For the cases of the spheres in the quaternionic and the Clifford algebras $\mathbb{R}_{(n)}$, by a generalized Fueter's theorem, Qian [4, 5] obtained a correspondence between a class of H^{∞}-Fourier multipliers and a class of holomorphic kernels, and proved that the corresponding class of H^{∞}-Fourier multipliers, the corresponding singular integral operators, and the induced Cauchy-Dunford functional calculus of the spherical Dirac operators are equivalent. Moreover, the mentioned operators are all bounded on $L^p(\Sigma)$. We note that, as necessary technical preparations of proving the correspondence and the boundedness of the operators, generalizations of the quaternionic Fueter theorem to the Clifford algebras $\mathrm{R}(n)$ were achieved: the cases n being odd were obtained by M. Sce [6], while the cases n being even were done by Qian [7], in the latter the fractional Laplace operators were defined via the corresponding Fourier multipliers. So far, the Fueter's theorem and its n-dimensional generalizations seem to be the unique approach to dealing with the singular integrals on Lipschitz perturbations of the spheres. The approach to analysis on the spheres offered by the Fueter theorem and its generalizations is an art of mathematics that may be viewed as the Clifford algebra version of "Starting from the unit disc" (see [8]). The further generalizations of Fueter's theorem and inverse Fueter's theorem have independent interest and applications; we refer the reader to [9].

This book establishes singular integral and Fourier multiplier theories in three different contexts: the Lipschitz curve context in the one complex variable setting; the graph type Lipschitz surface context; and the starlike Lipschitz surface context. The later two contexts are with the Clifford algebra setting. Chapters 1 and 2 are devoted to the theory of singular integrals and Fourier multipliers on Lispchitz curves. In Chap. 1, we state the boundedness, the singular integral expression, and the functional calculus of the Fourier multipliers. The analogous theory on the Lipschitz perturbations of the unit disc is given in Chap. 2.

In Chaps. 3–5, we will state systemically how to deal with the singular integrals and Fourier multipliers on the Lipschitz surfaces Σ by the technique of Clifford analysis. In Chap. 3, in order to make it self-containing, we state some basic facts and necessary backgrounds, including the Dirac operators, the Fourier transform, and monogenic functions on the sectors. At the same time, as a preliminary of the holomorphic Fourier multipliers on the Lipschitz surfaces, we introduce the generalizations of Fueter's theorem. In Chap. 4, we prove a Clifford martingale $T(b)$ theorem which implies the boundedness of the Cauchy-type singular integral operators. As is indicated above, for the main results of this chapter, there exists a parallel but different proof. We refer the interested reader to [1]. Chapter 5 includes the correspondence between H^{∞}-Fourier multipliers, the singular integral operators of monogenic kernels the Lipschitz surface Σ, and the H^{∞}-functional calculus of the spherical Dirac operator. The results of this chapter indicate that the Fourier multipliers and the monogenic kernels on the sectors play important roles in the theoretical framework of the Fourier multipliers and the singular integral operators.

The primary purpose of Chaps. 6–8 is to present the theory of the holomorphic Fourier multipliers on the starlike Lipschitz surfaces. In Chap. 6, we expatiate the results on the H^{∞}-Fourier multipliers on the starlike Lipschitz surfaces via the high-dimensional generalization of Fueter's theorem obtained in Chap. 3. In this chapter, we will give a detailed account of the estimate of the kernels of the operators with monogenic kernels. Chapter 7 is based on some new results on the fractional holomorphic Fourier multipliers on the starlike Lipschitz surfaces. The research on this topic is inextricably linked with the recent developments in the hyperbolic Clifford analysis. Theoretically speaking, the occurrence of the so-called "Photogenic Cauchy transform" implies that the study of the fractional Fourier multipliers is necessary. A well-known example of such class of Fourier multipliers is the fractional differential and integral operators with respect to the Dirac operator on the starlike Lipschitz surfaces. In addition, our study is significant for boundary values problems of differential operators on the starlike Lipschitz surfaces. In Chap. 8, using the complex analysis of several variables, we generalize the results of Chaps. 6 and 7 to the case of n-tours and the n-dimensional complex spheres. Particularly, the Cauchy-type singular integrals obtained by Gong [10] were extended to a family of singular integrals with holomorphic kernels. We also obtain the corresponding results of the fractional integrals and differentials.

In this book, we give a panorama-like and detailed demonstration of the theory of the holomorphic Fourier multipliers on the Lipschitz curves and surfaces. Through the writing of this book, we attempt to bring out the following core idea. Although the disposing technicalities vary with the different settings, the theories of different contexts all obey the same philosophy: the equivalence between the operator algebra of the singular integrals, Fourier multiplier operators, and the Cauchy-Dunford functional calculus of the Dirac operators.

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Contents

Nomenclature

$\mathbb{C}$	The complex plane
$\mathbb{C}^n$	The n-fold product of complex numbers
$\mathbb{C}_{(M)}$	The complex 2^M-dimensional Clifford algebra
$\mathbb{R}^n$	Euclidean n-spaces
$\mathbb{R}_{(M)}$	The real 2^M-dimensional Clifford algebra
$\{\mathcal{F}_m\}_{m=-\infty}^{\infty}$	A nondecreasing family of σ-field
$\{f_m^l\}_{m=-\infty}^{\infty}$	The left martingale generated by f
$B(x,R)$	Ball of radius R centered at x in $\mathbb{R}^n$
$C_{\omega,\pm}$	The heart-shaped regions on the complex plane
dx	Lebesgue measure
$E^l(E^r)$	The left (right) conditional expectation
$H^\infty\left(S_\omega^c\right)$	The class of H^∞ Fourier multipliers on sector $\left(S_\omega^c\right)$
$H^{p0}(\Delta)(H^{p0}(\Delta^c))$	The Hardy space on the bounded (unbounded) connected components of $\mathbb{R}_1^n \backslash \Sigma$
$H^s\left(S_{\omega,\pm}^c\right)\left(H^s\left(S_\omega^c\right)\right)$	The class of fractional holomorphic Fourier multipliers on $S_{\omega,\pm}^C\left(S_\omega^C\right)$
$K\left(H_\omega^c\right)$	The class of integral kernels related to H^∞ Fourier multipliers
$K^s\left(H_{\omega,\pm}^c\right)$	The class of integral kernels related to fractional holomorphic Fourier multipliers
$L^p(\gamma)$	The Lebesgue spaces on the curve γ
$L^p(\Sigma)$	Lebesgue spaces on the surface Σ
$N_\alpha^c(f)$	The exterior non-tangential maximal function of f
$N_\alpha(f)$	The interior non-tangential maximal function of f
$S_{\omega,\pm}$	The right- and left-half sector with the angle 2ω
S_ω	The sector with the angle ω
$W_{\omega,+}\left(W_{\omega,-}\right)$	The "W" (M)-shaped region on the complex plane