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P. P. Divakaran

The Mathematics of India

Concepts, Methods, Connections

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Preface

sadasajjñāsamudrāt samuddhṛtaṃ brahmaṇaḥ prasādena

sajjñānottamaratnaṃ mayā nimgnaṃ svamatināvā

(*Āryabhaṭīya, Golapāda* 49)

There is a good case to be made that the study of the traditional mathematical culture of India has now attained a level of maturity that allows us to begin thinking of going beyond just the description of its main achievements – theorems, constructions, algorithms, etc. – to attempts at a synthetic account of that tradition. Ideally, the scope of such a project should include not only the unifying mathematical ideas and techniques holding it all together, but also their place in the context of Indian history and Indian intellectual concerns. Its practical realisation will require many different skills and talents and is obviously a task for other times and other people. The more modest aim of the present book is to try and put together in a preliminary way what we already know from established sources, in the hope of drawing a first outline of its location within the universe of mathematics as well as, to a limited extent, in the Indian intellectual and cultural ethos.

The long and continuous history of the mathematical tradition of India was brought to an abrupt end in the 17th century. It was rediscovered – i.e., came to the notice of interested scholars outside the narrow circles of professional Indian astronomers (and astrologers) and mathematicians – some two centuries later. Many have contributed subsequently to the broadening and deepening of our knowledge and understanding of it. But areas of ignorance remained, of which the most serious and surprising was the astonishingly original and powerful mathematics created by Madhava and his long line of intellectual heirs in a small corner of Kerala, just before the arrival of European colonial powers on its shores. Recent work has begun to fill several such gaps and part of the aims of this book (as the footnotes will testify) is to give due weight to the results of the new scholarship. Two particular themes are worthy of special mention here. The first refers to what I have already invoked above: it is with a measure of disbelief that one realises that Madhava's name was largely unknown to modern historians of mathematics as late as thirty or forty years ago (and that the only translations of works by his followers, three in total so far, are not more than a decade old). So it is natural, for both historiographic and purely mathematical reasons, that I should pay particular attention to the work of the school Madhava founded and the attendant historical and social circumstances.

The other area of darkness – perhaps the most resistant to illumination – on which current research is beginning to throw new light touches on the quotation at the head of this preface (an English translation will be found in Chapter 6.4), namely the long-held notion that mathematicians in India arrived at their insights in mysterious ways that are different from how 'everyone else' did it. Aryabhata, as anyone who has read him should have known, had no doubts: the "best of gems" that is true knowledge can be brought up from "the ocean of true and false knowledge" only by the exercise of one's intellect, by means of "the boat of my own intelligence". Others have said the same in less poetic language. The first vindications of this ringing endorsement of the rational mind, through the reconstruction of the actual proofs and demonstrations, are another gain from the current analytical studies. I have thought it essential therefore to provide complete – though not always elaborate – proofs of most of the significant results in modern notation and terminology, either as direct transcriptions or by connecting together partial information available in different texts. The hope is that we can begin to chip away, little by little, at the encrusted layers of the mythology – and, occasionally, ill-informed bias – that still has some currency: that Indian mathematics was, in some strange and exotic way, not quite of the mainstream. If this book has an overriding message, it is that, conceptually and technically, the indigenous mathematics of India is in essence the same as of other mathematically advanced cultures – how can it be otherwise? – and that its quality is to be adjudged by the same criteria.

In navigating the ocean of true and false historical knowledge – an issue that has a special relevance in today's India – I have had (unlike, it goes without saying, Aryabhata) the generous help of many mentors, colleagues and friends. But before all of them must come the late K. V. Sarma whose work, more than anyone else's, helped place Madhava among the mathematical greats of all time, and not only of India: a fortuitous meeting with him while he was translating *Yuktibhāṣā* was the spark that first lit my interest in the subject. Among the many others who sustained it by sharing their expertise or in other ways are Ronojoy Adhikari, Kamaleswar Bhattacharya, Chandrashekhar Cowsik, Naresh Dadhich, S. G. Dani, Pierre-Sylvain Filliozat, Kavita Gangal, Ranjeev Misra, Vidyanand Nanjundaiah, Roddam Narasimha, T. Padmanabhan, A. J. Parameswaran, François Patte, Sitabhra Sinha, N. K. Sundareswaran, $\cdots$. A special word of appreciation and gratitude is due to Bhagyashree Bavare, my collaborator and consultant in matters Sanskritic, as well as to Samir Bose and N. G. (Desh) Deshpande who read the book in draft form and made many valuable suggestions. Above all, this book and I owe much to four old friends: S. M. (Kumar) Chitre for his faith and encouragement at a time when they were needed; David Mumford with whom I have had many stimulating exchanges about the content and context of the

mathematics of India; M. S. Narasimhan whose insights into what one may call the universal mathematical mind, freely shared over a long time now, have contributed greatly to my understanding of its Indian manifestation; and the late Frits Staal whose broad vision of India's intellectual heritage has strongly influenced my own thinking. Institutional support and hospitality at various times in the preparation and writing of the book came from the Institute of Mathematical Sciences in Chennai, the DAE/MU Centre of Excellence in the Basic Sciences in Mumbai, the Inter-University Centre for Astronomy and Astrophysics in Pune and the National Centre for Biological Sciences of the Tata Institute of Fundamental Research in Bengaluru. Finally, I am happy to acknowledge here my gratitude to the Homi Bhabha Fellowships Council for choosing me several years ago for a Senior Award, which event was the trigger that set off the idea of putting it all down on paper.

P.P. Divakaran
February 2018

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Introduction

0.1 Three Key Periods

This book has four parts. The first three correspond to chronological divisions of the long history of mathematical work done in India while the last part is a bringing together of the unifying ideas and techniques that run through that history. Two of our three historical periods are obligatory choices: the very beginning of mathematical thought in the Vedic age (earlier than ca. 500 BCE) as far as we can tell from available records, and the last phase (ending ca. 1600 CE) which is, naturally, much better documented. It is a remarkable fact that the end did not come as a slow fading away but, on the contrary, was relatively abrupt following a period of blazing glory; so it is doubly worthy of a detailed study. Between the beginning and the end is a middle or classical period spanning roughly the two centuries starting around 450 CE which, as it happens, is also of pivotal importance. This is the period of Aryabhata (Āryabhaṭa) and Brahmagupta, one in which the Vedic mathematical legacy was radically transformed.

As in all ancient mathematical cultures, so in India: mathematics began with learning how to enumerate discrete sets (counting) and how to create simple planar figures (drawing). The former took the form of decimal enumeration and led in due course to arithmetic and the latter to a fairly elaborate geometry, primarily of the circle and of figures associated with the circle. Historically, these first awakenings of the mathematical spirit span the Vedic age beginning, say, around 1,200 BCE if we go by textual evidence alone. There are, however, intriguing glimpses of an earlier familiarity with some of the Vedic geometry in the archaeological remains of the Indus Valley (or Harappan) civilisation, going back to the middle of the 3rd millennium BCE or a little earlier. What is certain is that the basic principles of both decimal arithmetic and geometry were pretty well in place at least by the time of the composition of the earliest of the architectural manuals called the *Śulbasūtra*, ca. 800 BCE, possibly somewhat before that. We do not have much information on the further development of geometry for a long time after, in fact until the classical period. But the consequences of the idea of measuring a number (I use 'number' without qualification to mean a positive integer), no matter how large, by means of the base 10, in other words decimal place-value enumeration, continued to be vigorously explored until the beginning of the common era and beyond. Historians are of the view that during this period, the Vedic people were in the process of settling the Gangetic plain, setting out from the upper Indus region and working their way gradually towards the east.

The classical period, which got its chief impetus from astronomical investigations, is characterised by several 'firsts'. To begin with, astronomy itself became a science: the motion of heavenly bodies came to be studied as the change in the geometry defined by them, considered as points in space, with the passage of time. The required geometry was seen to need a reformulation and refinement of that inherited from Vedic times. Simultaneously, the reconciliation of time measurements made using the lunar and solar periods led to problems requiring the solution of algebraic equations in integers (and of course, first, a proper algebraic formulation of such equations and their properties). Circle geometry became greatly sophisticated, going beyond the needs of astronomy. Also from this phase we have the first texts concerning themselves exclusively with mathematics and astronomy which are, moreover, attributable to historically identifiable authors: Aryabhata first and foremost, followed a little over a century later by Brahmagupta and Bhaskara (Bhāskara) I. For those who know even a little about the story of mathematics in India, the mention of these names is enough to indicate the significance of this period for everything that came later. Indeed it is not an exaggeration to say that all of the subsequent work in astronomy and related mathematics in India is an elaboration of the concepts and techniques first formulated during this time.

The geographical locus of mathematical activity during this middle phase was fairly widely distributed within northern India; Aryabhata did his work in Magadha, part of modern Bihar, while Brahmagupta is thought to have lived in Malava (Mālava), the region around Ujjain (Ujjayini).

My third chronological division corresponds also to a geographical division and it is defined entirely by the achievements of Madhava (Mādhava). Madhava and a long line of his disciples lived and worked in the lower basin of the river Nila (Niḷā) in central Kerala and they remained productive for two centuries, ca. 1,400 - 1,600 CE. It is by far the best documented phase of Indian mathematical history. Despite that, it has taken a long time for modern scholars to go from relative ignorance to puzzled admiration to an informed appreciation of the brilliance and originality of the achievements of this last phase.

The central concern of the Nila school[1] was the conceptualisation and execution of a method of dealing exactly with geometrical relationships in nonlinear problems, specifically problems of rectification and quadrature involving circles

[1]The name is meant to convey not only the thematic coherence of the mathematics it produced but also its localisation in a cluster of villages all within easy walking distance of each other. Other designations such as the Kerala school and the Aryabhata school are used by some authors. The first ignores the fact that there was a shortlived but vibrant astronomical research centre elsewhere in Kerala in the 9th century CE. As for invoking Aryabhata's name to describe their work, almost all of Indian mathematics from the 6th century onwards can with justice be called the Aryabhata school.

and spheres, by resorting to a process of 'infinitesimalisation' through division by unboundedly large numbers. In other words, the fundamental achievement of Madhava and the Nila school was the invention of calculus for trigonometric functions (as well as, along the way, for polynomials, rational functions and power series). The roots of the Nila work go directly back to Aryabhata, skipping over much of what happened in between, and, in a very real sense, it represents the coming together of the two main strands of the earlier tradition, geometry (of the circle and associated linear figures) and based (decimal) numbers. In the process, it sharpened and deepened algebraic methods traceable in a formal sense to Brahmagupta, defined and made respectable entirely new types of mathematical objects (infinite series) and introduced methods of proof (mathematical induction) unknown till then.

One would expect a time of such effervescence and creativity to be followed by a period of consolidation and steady progress. But that is not what happened. The Nila school marks the final episode in the long and essentially autonomous progression of mathematical thought in India. Little of any value or novelty emerged after the 16th century in Kerala or indeed in India as a whole. Quite a few reasons have been put forward for the decline – it is not every century that produces a Madhava – but it cannot be doubted that the immediate trigger was the arrival of Portuguese colonialism on the shores of Kerala.

The choice of the three epochs that we are going to concentrate on is thus not random. Historically and in terms of mathematical impact, they represent the highest of high points in a long evolution. What of the gaps – each, coincidentally, of about a thousand years, give or take a century or two – that separate them? By and large, they served as periods of steady if unspectacular advances. There were of course very many fine mathematicians and astronomers, especially between the middle and the final phases, who do not fit into our overneat division, the most outstanding being the versatile and prolific Bhaskara II. Their contributions were handsomely acknowledged by those who came later and will figure in our account – Bhaskara II in particular had an enormous influence and his *Līlāvatī* is probably the most popular mathematical text ever written in India. The hope is that what emerges at the end is a well-rounded portrait of mathematics within the boundaries of cultural India – which rarely coincided with the various political Indias of kings and empires – and over its recorded history. The informed reader will judge how well the hope is realised.

In the short concluding part I have tried to flesh out the sense of continuity and unity in the mathematical culture of India, already visible to the enquiring eye in the first three parts, by gathering together some of the common strands of thought and method that run through it all. They relate to foundational issues as well as to the specific means employed to get specific results - in fact the two aspects are so inextricably joined that they appear to be different expressions of an all-pervasive common mode of thought. The prime example of this unvarying 'mathematical DNA' is from geometry which is almost always reduced, from the *Śulbasūtra* down to the Nila texts, to the application of two principles: the

Pythagorean property of general (i.e., not restricted to integral or rational) right triangles and the proportionality of the sides of similar (generally, right) triangles. Similarly, the recursive principle underpinning the construction of decimal place-value numbers and the early realisation that they are unbounded have big roles to play in the treatment, 2,500 years later, of power series in the Nila work. Even the inductive proofs which make their first appearance in this last phase are introduced from a recursive standpoint, as distinct from the formal-logical foundation favoured later in Europe.

Just as conspicuous as these commonalities is the absence of concepts and methods which the modern mathematically literate reader accepts unquestioningly. For instance, though the operation of division and the notion of divisibility are present from very early times – the place-value construction of numbers is based, after all, on the iterated application of the division algorithm – there appears to have been no special significance attached to prime numbers, and it is not clear why. Less puzzling is the avoidance of the powerful and ubiquitous method of proof based on the rule of the excluded middle, proof by contradiction. The logical basis of reasoning by contradiction was considered by philosophers from a very early time, pronounced unsound and rejected as a means for the validation of knowledge. A belief used to be prevalent in some mathematical-historical circles (and is occasionally still expressed) that Indian mathematicians did not have proofs for many of their assertions or, worse, that a conception of a logical proof itself was absent. It will be seen that this is a myth; only, the logic is somewhat different from what is axiomatic in the European approach, that of a progression from axiom to definition to theorem using, at every stage, equally axiomatic rules of reasoning.

0.2 Sources

The term 'India' or, occasionally, 'cultural India' as used in this book encompasses of course political India at the time of independence but also covers what is called the Indian subcontinent, extending well into Afghanistan in the northwest. Historically and culturally it also included large parts of southeast Asia which, as will be seen, made at least one significant contribution to the reconstruction of the whole story.

The recorded evidence on which the reconstruction must be based comprises any surviving material, concrete or (more or less) abstract, from which we can draw reasonably secure conclusions about those preoccupations of the past inhabitants of India that have a clear mathematical content. The vast majority of such records are texts dealing with mathematics, generally in parallel with applications to some practical science which, in later times, was almost invariably astronomy. But there are other sources of information as well which supplement the texts and make up for their shortfalls. An extreme instance is the earliest historical period, the Indus Valley period, which has left behind a splendid collection of ruins but whose writing has not yet been deciphered.

The only recourse we have then is to try and 'read' civilisational artefacts other than texts, abstractions such as townscapes and street plans, floor plans and elevations of buildings, geometric decorations on seals, pottery, etc., as well as material objects like weights and other measures. While little can be said with absolute certainty, surprisingly good information of a general mathematical nature can be plausibly extracted from Indus Valley material of various sorts.

Textual sources can be divided into two types to begin with, those whose mathematical content is scattered and incidental and those devoted wholly to expositions of mathematics (and its applications). Most of the early texts of either type were composed orally, memorised and, in the early part of their existence, transmitted orally before being transcribed on perishable material. But before turning to these two types, I should mention a different kind of written evidence, inscribed permanently (very permanently) on stone monuments, coins, copper plates recording grants etc., the most famous being the first symbolic representations of zero as an approximate circle carved on stone (southeast Asia and Gwalior in India). They are comparatively late, the earliest numerical inscriptions (outside the Indus Valley seals) being from the centuries around the turn of the common era, showing numerals in the Brahmi (Brāhmī) script. The late occurrence of written numerals is consistent with the primacy of number names rather than number symbols in India, a topic which will occupy us later. After they came into common use, Brahmi numerals seem to have gone through a complicated evolution before settling down to the currently standard Arabic numerals as well as various minor variants in Indian languages. They are well documented in several studies, notably in the book of Datta and Singh ([DS]).

The texts I have characterised as incidentally mathematical are of paramount importance as they are, with the exception of the *Śulbasūtra*, our only primary source of knowledge about mathematics in the Vedic period, in particular about the genesis and rapid development of decimal place-value enumeration and its use in elementary arithmetic. They comprise, first, the earliest Indian literary productions, namely two of the original Vedic (*saṃhitā*) texts, the *Ṛgveda* and the *Yajurveda*, the latter more particularly in the earlier of its two recensions, named *Taittirīya Saṃhitā*. Some useful information also comes from a few of the *Brāḥmaṇa*s and *Upaniṣad*s which are, by and large, exegetic elaborations of Vedic ritual and philosophy. Scholars date the gathering together of the *Ṛgveda* into ten Books or 'circles' to around 1200 - 1100 BCE but also think that the individual poems or hymns were composed in the centuries leading to that period from the time of the appearace of the Vedic people in northwest India. The *Taittirīya Saṃhitā*, which is mostly a text describing details of ritual practices, is probably from only a little later and is full of various regular sequences of numbers, including lists of names of powers of 10. It would seem from the way these lists are presented that the first intimations of the unboundedness of numbers were already in the air. The fascination with the potential infinitude of numbers continued for a long time – perhaps not so long considering that the quarry being chased was unattainable – leading to the naming of sequential powers of 10 extending to mind-boggling magnitudes. Evidence for this comes from religious texts (Vedic/Hindu as well as Jaina and Buddhist) and from other sources like the great epics *Rāmāyaṇa* and *Mahābhārata*, composed in the form we know them today probably just before the common era.

The value of literary and religious texts as repositories of mathematical information is greatly enhanced by the surprising – for a people who turned everything they knew into literary compositions – absence of any work that lays out the principles governing decimal enumeration and decimal arithmetic. That is not the case when it comes to early geometry. The *Śulbasūtras* are traditionally said to have existed in a number of versions of which four appear to have survived more or less intact. They are thought to be different recensions of material drawn from a common store of knowledge, made at different times (and perhaps in different places in the Gangetic basin) ranging over ca. 800 - 400 BCE. Being handbooks for the construction of Vedic ritual altars, their purpose is primarily architectural but the precise and quantitative treatment of floor plans having different regular shapes turns them effectively into workbooks of plane geometry. In this respect the *Śulbasūtra*s stand apart from all the other sources we have for mathematics in the early period.

The middle or classical period brought about a radical and permanent change in the nature of mathematical enquiry and, with it, in the character of the texts. Astronomy became the driving force of mathematics; in fact the two sciences became inseparably linked. Going by the texts, all astronomers were also mathematicians and most mathematicians were astronomers, a trend that survived right until the last phase. It is tempting to say that most of them would have thought of mathematics as the prerequisite for doing astronomy rather than of astronomy as applied mathematics. (From now on it will be understood that the word 'mathematician' as used in this book will refer generally, with one or two obvious exceptions, to an astronomer-mathematician.)

As for their texts, I have already noted that the classical period saw the appearance of what we may call monographs for the first time (the *Śulbasūtra*s partially excepted). Their production became increasingly frequent and one has the sense that this period also marks the emergence of a professional mathematical community.

The texts themselves can be divided loosely into two broad categories: original works having some new mathematics to expound and commentaries (*bhāṣya* or *vyākhyā*; there are also other less common names) on them (and an occasional commentary on a commentary). The number of commentaries that a particular work generated seems to have been an increasing function both of its significance (and, hence, of the esteem and reverence in which its author was held) and of its brevity (and, hence, of its difficulty). The prime example is of course *Āryabhaṭīya*, highly esteemed and very short, with about half a dozen known commentators extending from Bhaskara I (early 7th century) to Nilakantha (Nīlakaṇṭha) (early 16th century) trying their hand at making its cryptic verses more accessible; indeed, it is only a mild exaggeration to characterise most of the work during this time as constituting a working out – a multi-author *mahābhāṣya* so to say – of Aryabhata. Other influential examples are *Brāhmasphuṭasiddhānta* of Brahmagupta, some of the writings of Bhaskara II and, from the late phase, Nilakantha's *Tantrasaṃgraha* which inspired two very important self-described commentaries on it in less than 50 years. Naturally, the commentaries also tend to be longer; and they are often in prose, more frequently than the source texts to which they are anchored.

Nonetheless, the division into these two categories is not watertight. Firstly, every text starts with a résumé of the knowledge already acquired, whose purpose was to serve as the platform on which new mathematics will be built – even *Āryabhaṭīya*, possibly the shortest mathematical treatise in history, begins with an evocation of decimal enumeration by recalling the names of powers of 10 as well as some elementary geometry. For the historian this brings in two advantages: filtering out (the very infrequent) patent errors from earlier work while at the same time giving a glimpse of how later mathematicians understood (or, occasionally, misunderstood) the material they were building on. We shall meet instances of both in the course of this book.

Conversely, many commentaries – so described in their titles or opening passages – not only provide clarifications and interpretations of the original work, but also put forward genuinely new ideas and insights. A very good example is Nilakantha's *Āryabhaṭīyabhāṣya* which subjects Aryabhata's text to a searching analysis and reinterpretation in the light of the new results of Madhava and the Nila school and, in the process, turns some of the imprecisely worded aphorisms in the original into precise mathematical statements which Aryabhata himself may or may not have had in mind.

The generally accepted explanation for most of these differences of presentation is that the original texts were intended to be memorised as the core of a guru's teaching, to be supplemented by face-to-face explanations and demonstrations. In any case, no modern reading of an ancient mathematical classic can be considered complete without attention being paid to the more important commentaries that came later, even in the rare cases where they may have been misled.

The language of Indian mathematics is, overwhelmingly, Sanskrit. From Vedic times until the very end, from the northwest to the extreme south and at all places in between, all the main texts – with one very significant exception – were composed and communicated in the variant of Sanskrit prevalent at the particular time, without much geographical variation. During the first period when Vedic Sanskrit was the natural language of literate people and the second period when classical Sanskrit was the language of learning in north India, this is to be expected. As mainstream mathematics spread further south to regions with a strong local literary and linguistic life of their own, the habit of doing traditional sciences, the *śāstra*, in Sanskrit survived. An explanation for this conservatism may be that Brahmins, who were the people of learning and who began migrating to south India in large numbers in the second half of the first millennium CE, remained protective of their inheritance for a long time. Apart from some historical evidence that this was so, quite strong in the case of Brahmins in Kerala, it is a fact that the style of formal Sanskrit mathematical presentation hardly changed from Aryabhata to Nilakantha.

The significant exception mentioned above is the remarkable treatise *Yuktibhāṣā*, written in Kerala around 1525 CE by a "venerable twice-born" (Brahmin) named Jyesthadeva (Jyeṣṭhadeva). It is remarkable for many reasons as we shall see – a large part of Part 3 of the present book depends critically on its reading – but what is to be noted here is that it is not in Sanskrit but in Malayalam, the language of Kerala. And it is not as though it marked the beginning of a trend; several works were composed in Kerala during the following century in chaste classical Sanskrit. We have no idea why Jyeshthadeva decided to popularise, if that was the intention, a science which until then had remained the preserve of those who were proficient in Sanskrit.

Another thing we do not know is when the texts began to be written down. There is little doubt that material from the Vedic age was communicated orally for a long time. Even in the classical age, by which time writing in various scripts had become fairly standard, oral teaching and learning still retained its preeminence. Nevertheless it is a tacit assumption in Indological studies, very likely correct, that most material got written down sooner or later. (I will use terms like 'writings' and 'books' to refer to work which may have begun their life orally, as I have used 'literate' to mean also orally literate beyond the needs of basic functional communication). In any case, what was actually written down was on perishable material like palm leaves and manuscripts had to be copied and recopied periodically for survival. Consistent with the history of the influx of Brahmins into parts of south India, most mathematical manuscripts originating in northern India (of the classical age and later) were unearthed in Kerala, the last refuge of the sciences in colonial India, often from the private libraries of Brahmin families. At the same time, the impact of Sanskrit on the local language gave rise to modern Malayalam, written in a rapidly evolving script of its own. At some point in this process of sanskritisation, Sanskrit itself came to be written in the local script, with the end result that the vast majority of surviving Sanskrit manuscripts in Kerala – i.e., a large proportion of *all* surviving manuscripts – are in Malayalam script. It will obviously be absurd to conclude from this, as some writers still do, that Aryabhata for example was born in Kerala.

In the light of all this, it is natural to wonder whether we have in hand today the essential primary sources necessary for the delineation of a satisfactory picture of the totality of Indian mathematical culture, in other words whether some critically important manuscript may not still be lying unsuspected in the recesses of some old family library or lost permanently to the ravages of time. We cannot of course be sure but, here again, the persistence of knowledge acquired earlier across generations of later writings provides a guarantee, I think, that nothing truly significant has disappeared for good. There was a time in the first half of the 19th century when scholars had access to Brahmagupta's writings with their numerous and hostile criticisms of Aryabhatan doctrine, but no first-hand knowledge of the *Āryabhaṭīya* itself. The world was saved the need to guess what the *Āryabhaṭīya* really contained by its publication in 1874. The nearest to finding oneself in a similar predicament again is the absence of any known mathematical text – a fragmentary astronomical work of little mathematical interest apart – attributable to Madhava. We can always hope that such a manuscript will one day come miraculously to light but that is unlikely to change our understanding of his mathematcs; the extensive writings of those who followed him in the Nila school already give us a very coherent idea of what it might contain.

There are also issues on which the modern reader would prefer to get a little more help than is available in the texts. The first concerns diagrams. Perhaps because drawing with a heavy iron stylus on dried palm leaf is not easy, they are absent from manuscripts except for an occasional rudimentary sketch. This can be a handicap when trying to follow a complex line of geometric reasoning (examples are Brahmagupta's theorems on cyclic quadrilaterals and the infinitesimal geometry of the Nila school). What there is instead are descriptions of the necessary figures, which are generally adequate. In face-to-face instruction, diagrams were presumably drawn on sand covering a board or the floor, a practice which was prevalent in Kerala until as late as the middle of the 20th century.

The other striking absence is that of a symbolic notation for mathematical objects and operations on them even where long and intricate and fundamentally algebraic reasoning is involved, as in Brahmagupta's treatment of the quadratic Diophantine equation (later called Pell's equation in Europe) or in the derivation of trigonometric infinite series by the Nila school. It is possible that this is a legacy of the oral tradition, but it is a fact, nevertheless, that abstract representations of linguistic objects and sets of such objects were already employed by Panini (Pāṇini) (ca. 5th century BCE) in his treatise on grammar. In mathematics, the use of syllables to stand for unknown numerical quantities was initiated by Brahmagupta and given much prominence by Bhaskara II in his monograph on algebra (*Bījagaṇita*). But one will search in vain for a symbolically expressed equation. In consequence, a considerable part of the effort of reading a text has to go into turning narrative descriptions into the present day mathematical language of symbols and symbolic statements of relations among them. Far more seriously, the disdain for a symbolic presentation may be a reflection of a certain methodological position which in turn led to a reluctance to see mathematical truth in full generality and to obvious paths not taken. We will return to this point in somewhat greater detail in Part IV.

On the other hand, the absence of detailed proofs in many of the texts, whether originals or commentaries on them, is not a serious matter. With very few exceptions, proofs of stated results, either complete or detailed enough for an informed reader, can be found in some text or the other. We have come to see recently that even in accounts in which the logical development towards a final statement ('theorem') is supposed not given, a proper understanding of the fine distinctions in the terminology employed may in itself be a good guide to the reconstruction of the logic. More generally, the way a proof – the commonly used word is *upapatti* or, in later times, *yukti*, 'reasoned justification' – is put together and what it assumes as self-evident rules of reasoning are quite different from the modern approach to such questions. Logicians throughout history have wrestled with the issue of what constitutes a good criterion for the validation of new knowledge. In this book I use the word 'proof' without further qualification in the sense accepted by Indian mathematicians – we shall encounter numerous examples below – even though that may not meet with the approval of axiomatists ancient or modern; Indians did not believe in axioms. It would seem that just as Indian mathematical authors have their own particular style of presenting their knowledge, so does it require a particular way of reading to absorb it in full measure and to be convinced by it.

This short general survey of the sources will be incomplete without a look at one written work to which much of the general discusion above applies with a high degree of reservation. The Bakhshali manuscript as it is known was unearthed (literally) in 1882 in what is now Pakistan, within or close to the borders of ancient Gandhara. It is written on birch bark, a popular medium of writing from at least the beginning of the common era in the regions of north India where the Himalayan birch grows. The fraction – how small a fraction remains unknown – of the complete text that has survived is in such a fragile condition as to preclude scientific methods of dating (in the judgement of its keepers). Whether it is a copy or a recopy of a preexisting work or the original is also not known. In the latter case, the lack of a scientifically determined date is especially unfortunate because its contents are in many respects singularly novel. As it is, based on its mathematics, its language (a local version or dialect of Sanskrit) and script (also a local variant of the Brahmi/Devanagari family), not to mention personal preferences, people have dated it anywhere from the 2nd century BCE to the 13th century CE. Even if we discount the extremes of this quite impossibly wide range and date its substance between, say, the 4th and the 8th centuries CE, the historical conclusions we can draw from it will depend dramatically on precisely when it was composed since these centuries encompass the transition to the classical age and a very rapid evolution within that period. In particular, an early, say pre-Aryabhatan, date – I will in fact argue later for just such a position – will force a radical revision of our present understanding of the evolution of arithmetical and algebraic ideas in the gap between our first two periods.

A quick look at the contents of Bakhshali shows us how and why. It is almost unique among the standard texts in utilising, at several levels, a fairly evolved symbolic notation. First, numbers are written in symbols (as distinct from their names) in decimal place-value notation using a variant of Brahmi numbers. (The earliest Brahmi numbers on inscriptions etc. that I mentioned earlier did not follow a strict place-value notation). Then, arithmetical operations were also represented abstractly (often in addition to a narrative description) and that encompassed both negative numbers and fractions as well as schemata for equations. As cannot be avoided in a written symbolic notation, there is a symbol (a dot) for zero. And finally, there are algebraic equations with a symbol representing the unknown. These features signalling an evolved arithmetical culture would indicate a relatively late date for Bakhshali if we insist on dated, independent corroboration, e.g., on Brahmagupta for the algebra of positives and negatives or for the use of syllables for unknowns. But if Bakhshali itself could be reliably dated to be early within its plausible range of dates, that would push back the time of the first signals of an algebraic mode of thinking by a few centuries and thereby give us a measure of the progress made in the interregnum between the end of the Vedic phase and the beginning of the classical phase. Such a chronology is not out of the question; the *Śulbasūtra*s do contain some arithmetic with fractions and subtractions (negatives?). Notational devices like those in Bakhshali resurface in one or two later texts – the great classical works avoid them – lending support to the view that they were in continual use behind the scenes, in traditional teaching, in the classroom as it were.

The decline in creative mathematical activity in Kerala (and India as a whole) that followed the arrival of the Portuguese caused a corresponding decrease in the number of new texts as well. It is true that the 16th century saw the production of some masterpieces, *Yuktibhāṣā* among them, but, with one or two exceptions, there was not a great deal that was original, even in terms of interpretation, in the few serious books that were written after the 16th century. What did happen instead, beginning in the late 18th century, was a slow rediscovery of traditional Indian learning, including mathematics and astronomy. Naturally, the pioneers in this endeavour were British Sanskritists, following in the footsteps of the first and the finest of them, William Jones of Calcutta. Towards the end of the 18th century and in the first few decades of the 19th came the first translations, notices, commentaries and lectures (the translation of a part of *Sūryasiddhānta* by Samuel Davis and John Playfair's essays based on it, Colebrooke on Brahmagupta and Bhaskara II, Whish on some Nila texts, and several others). The second half of the 19th century saw a gradual acceleration of this activity in a number of European countries and even in America, bringing several key texts to the notice of scholars, *Śulbasūtra* and *Āryabhaṭīya* among others. Thus came about the slow and hesitant process by which the world came to recognise the existence of an Indian culture of mathematical enquiry that was in large measure self-motivated and self-sufficient.

While a residual knowledge of traditional texts and their utilisation in preparing almanacs survived through colonial times, the reawakening of a critical and informed interest in their own true mathematical heritage among Indian scholars came only towards the end of the 19th century. It has grown vigorously since. Given also the continuing activity in other parts of the world in bringing out these riches that long lay hidden, this seems a propitious time to attempt an account of mathematics in India – the mathematics of India – that not only explains its high achievements but also pays attention to its own special characteristics, its Indian identity as it were. It goes without saying that any such attempt must rely on the work of the fine line of scholars, extending over more than two centuries now, who have brought their linguistic and mathematical skills to the elucidation of what, for many of them, is an alien way of doing and presenting mathematics. The informed reader will find many instances in this book of my indebtedness to the insights of these secondary sources. It is astonishing to realise, nevertheless, that there have only been three books ever that have taken on the task of presenting a wide-ranging and mathematically oriented account of the subject. Of these, the otherwise encyclopaedic two-volume book of Datta and Singh ([DS]) published in the 1930s omits geometry (the promised third volume never came out except, much later, as a posthumous journal article), but Sarasvati Amma's comprehensive and magisterial book ([SA]) – faithful and lucid at the same time – more than compensates for the omission. The very recent book of Kim Plofker ([Pl]) covers most of the geography and history of Indian mathematics and has an extensive bibliography of current research. All three books are essential reading.

0.3 Methodology

The present book differs from those referred to above in several respects. Firstly, it is not as comprehensive; instead, the premise here is that by concentrating on the three key periods, supplemented by a broader account of the relevant material from the periods in between, we can arrive at a faithful portrait of Indian mathematics as it became deeper and broader with time. Within this general framework, certain themes relating to the posing and solving of important problems and/or departures along new directions can be identified. These milestones, together with an indication of how earlier work fed into them, are shown in the schematic diagram alongside. Consistent with this thematic partition, the approach followed in the book is largely chronological. But there are places where I have had to deviate from strict chronological progression, partly in order to accommodate proofs which are often found in later commentaries, even when we have reason to believe that they were known earlier. More generally, it helps to highlight the continuing relevance of certain themes across very long time spans, for example the influence of the principles of decimal enumeration on several aspects of the work of the Nila school (see the flow diagram). Also, occasionally, I have found it more illuminating to look at the initial steps in the study of a particular topic from the vantage point of how it developed later. An example, out of many, of the advantage of inverting the chronological order is the light Vedic geometry throws on its possible Indus Valley antecedents.

It is not possible in a book of reasonable size to treat every advance in all these areas with equal thoroughness. Choices have had to be made – there was a lot of mathematics in ancient India, and many mathematical texts. As a rule, I have tried to pick for detailed treatment those topics which are central

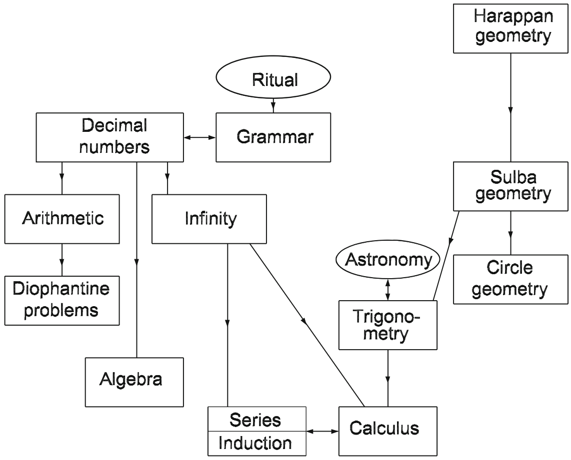

Figure 1: The main themes. Time runs from top to bottom, though not to scale. Oval compartments are for topics not discussed in detail. 'Infinity' is shorthand for the unboundedness of numbers and 'circle geometry' stands for the geometry of cyclic figures, especially quadrilaterals. Combinatorics is missing from the diagram; it derived from prosody and had little direct impact on later mathematics. The attentive reader will probably add a few extra arrows to the diagram.

to the overall development depicted in the flow diagram, the 'big themes': decimal enumeration in the early Vedic texts, Aryabhatan trigonometry, and the invention of calculus by Madhava, to mention only the most prominent. I have also paid special attention to recent work either addressing questions which had not been studied earlier or clarifying issues left ambiguous or unresolved in previous work. Typical examples would be the use of the geometry of intersecting circles in making squares and generating space-filling periodic patterns (Indus Valley), the logic behind Brahmagupta's theorems on cyclic quadrilaterals and the first glimpses of an abstract algebraic mode of thinking in the introduction of general polynomials and rational functions in one variable in the Nila work. Given the interdependence of these themes, it goes without saying that there will be a fair degree of overlap, much to-ing and fro-ing, in their treatment. For the same reason – or due perhaps to the influence of the admirable pedagogic techniques of *Yuktibhāṣā*? – there will also be a certain amount of repetition of some of the key points and themes.

Secondly, as far as the historical aspects are concerned, the approach adopted here is somewhat different from the usual. Due to the efforts of many scholars, we know a little more than we used to, not so long ago, about the lives and work of quite a few of the mathematicians belonging to historical times – basic biographical data, approximate dates and places of composition of key texts and so on – though much remains to be discovered in future work.

Since these facts will (with a few very important exceptions) only get a passing mention, it may be useful to the non-specialist reader to see summarised in one place what is known of the chronology of the major landmarks and their authors, together with the key texts. As far as the early phase is concerned, that is easily done: we have only to remember that the *Ṛgveda* was compiled in the form in which we know it around the 12th century BCE from material composed earlier by anonymous poets (even if names can sometimes be attached to particular poems), the earliest *Śulbasūtra*s (of Baudhayana and Apastamba) in the 8th century BCE and the *Taittirīyasaṃhitā* roughly halfway between the two. Further down the line, it seems safe to place Panini in the 5th century BCE and Pingala a century or two afterwards. All these dates, based mostly on indirect linguistic evidence, reflect the consensus of modern scholarship.

From Aryabhata onwards, we are on much firmer ground. The following list, confined to those who had the greatest impact, either on their followers or on the judgement of modern historians, is only meant as a first orientation. (I have omitted Bakhshali since we know neither the date nor the authorship).

Aryabhata (born 476 CE): His only known work, *Āryabhaṭīya*, is internally dated 499 CE.

Bhaskara I (late 6th - early 7th century): The first prophet of Aryabhata, author of three expository works, among them *Āryabhaṭīyabhāṣya*. All of them had great influence on the spread of the Aryabhatan doctrine.

Brahmagupta (first half of the 7th century): Author of *Brāhmasphuṭasiddhānta* (628 CE), strongly anti-Aryabhata on details of doctrine, and of *Khaṇḍakhādyaka* (about 650 CE), in which he reconciles his astronomical model with that of “master” (*ācārya*) Aryabhata.

Śaṅkaranārāyaṇa and his teacher Govindasvāmi (9th century): The first astronomer-mathematicians authentically established to have worked in Kerala, devoted followers of Aryabhata and Bhaskara I, and precursors of the Nila school. *Laghubhāskarīyavivaraṇa* of the former is dated 869 CE.

Mahāvīra (9th century): His book *Gaṇitasārasaṃgraha* has no astronomy but is strong in arithmetic and mensuration. The only historically identifiable Jaina mathematician and the least Aryabhatan of all.

Bhaskara II or Bhāskarācārya (born 1114): Master of the astronomy and mathematics of his time, a great teacher and consolidator. His *Siddhāntaśiromaṇi* (mid-12th century) incorporates *Līlāvatī* and *Bījagaṇita*.

Narayana (Nārāyaṇa Paṇḍita, 14th century): Accomplished in all aspects of mathematics. His *Gaṇitakaumudī* (1356) has many strikingly original results, especially in combinatorics and cyclic geometry. May have been from Karnataka (or, remotely possibly, from Kerala) which will, perhaps, link him to

Madhava (approximately 1350 - 1420 ?): Astonishingly little is known about the inventor of calculus. One short text attributed to him, on the moon's motion and, maybe, one or two other fragments.

Nilakantha (1444 - 1525-1530): The conscience-keeper of the Nila lineage. Wrote many books of which *Tantrasaṃgraha* had great influence. The masterful *Āryabhaṭīyabhāṣya*, which he himself calls *Mahābhāṣya*, the Great Commentary (like Patanjali's commentary on Panini's *Aṣṭādhyāyī*), written late in his life, is *the* key to Aryabhata's mind.

Jyeshthadeva (second half of 15th - first half of 16th century): Hardly anything known about his life. Disciple of Nilakantha and author of *Yuktibhāṣā*, by far the most important book about the work of Madhava and the Nila school (and, in my view, one of the classics of all time).

The overall picture that emerges is of a coherent mathematcal culture, no matter who contributed to what parts of it, or when, or where. I will return to the theme of the sense of unity all of them together convey at the end (Chapter 14) of this book.

A third issue which is really subsidiary to the main concern of this book but which cannot be avoided is that of possible mathematical contacts between India and other cultures at various times. It cannot be avoided because there has been a trend among historians in the European tradition, from Colebrooke onwards and still quite alive, to hypothesise external influences – Babylonian, Greek, Chinese, etc. – in the genesis of several Indian mathematical developments. This is a difficult issue to be categorical about. Documentary or other unimpeachable evidence in support of (or against) such transmissions does not exist as a rule. Historical plausibility based on cultural contacts at the appropriate time can sometimes be suggested but, often, cannot be built upon. In these circumstances it would seem reasonable that the case for or against transmission is best made by looking at the parallel evolution of a mathematical theme or a related set of themes as a whole, as well as at ideas and techniques that are *not* held in common, rather than at the odd shared factoid, in other words by paying attention to the totality of the internal evidence. On the occasions when this issue does come up (mostly, but not only, in relation to geometry), I have tried to do that. A good case study is the clear and strong Alexandrian, specifically Ptolemaic, influence on Aryabhata's system of astronomy. The internal evidence for a corresponding inflow of purely mathematical ideas is weak and impossible to identify objectively, despite some confident claims made to the contrary. It is perhaps inevitable that, in the reverse direction, there is currently a tendency among a few writers to see European calculus of the 17th century as deriving its initial impetus from the earlier work of the Nila school. Here again, there is so far no evidence in support of the hypothesis. This too is a question I will discuss as it comes up and at the end (Chapter 16.2).

If evidence of a flow of mathematical ideas into India is weak, the case for an organic evolution within India, dating back as far as we can go, is correspondingly strong; the delineation of that continuity is one of the aims of this book. As part of that aim, I have tried to frame Indian mathematics within the general context of Indian history and culture, of the movement of people and ideas, linkages with other disciplines, in short the intellectual climate of the times. The historical interludes that are occasionally sandwiched among the descriptions of the mathematics are meant to serve that purpose. Apart from providing the general context, it is a fact, at least in India, that a broader cultural view often throws a direct light on the substance and style of the mathematics itself, though it is understood that culture and history alone never produced a genius. Like all historical reconstructions, such an endeavour cannot be anything like the last word; new facts and fresh insights will continue to emerge and bring with them the need to reevaluate some of the connections tentatively suggested here.

That having been said, it bears repetition that this book is really and fundamentally about the mathematics. In particular, even astronomy *per se*, which was the original motivating force for the Aryabhatan revolution, will only get passing mentions.

In the manner of presenting the mathematical content, the historian is faced with the eternal dilemma of all history of science: finding the right balance between literal faithfulness to the sources and the wish to see what was done in the distant past from the perspective of what we consider interesting and important today. Absolute fidelity to the original texts risks losing the present day reader, used as he or she is to the habits of thought and discourse of a science, mathematics in the present case, that has grown unrecognisably in depth, generality and sophistication in the last few centuries. It is of course axiomatic that a literal reading of texts must remain the foundation of whatever conclusions one may draw from them and good translations have precisely that function, leaving the reader to make the adjustments necessary to fit these into the mathematical universe that we inhabit now. For European mathematics, this has been relatively easy to manage as the modern style of mathematical writing is in a direct line of descent from the Mediterranean-Greek. There are in fact successful books which rely on that approach. Such total literalness is less likely to succeed with Indian material, which has its own style of communication – as already noted in the section on sources – indeed its own way of thinking of and doing mathematics. Most scholars of the Indian tradition, from Colebrooke down, have understood this, at least implicitly. Several recent studies have in fact taken the route of accompanying a translation of the work concerned, also given alongside in the original language, with detailed notes in the modern style and in modern terminology and notation: Sarma (*Yuktibhāṣā*), Patte (commentaries on *Līlāvatī* and *Bījagaṇita*), Ramasubramanian and Sriram (*Tantrasaṃgraha*), to cite only the most recent.

In a general book such as this, aimed mainly at interested but not specialist readers, this scholarly ideal is difficult to sustain even for selected extracts and perhaps not desirable. In its place, I have adopted a less demanding approach, limiting direct citations from the texts to translations of a few (very few) essential or insightful passages and freely using symbols, equations, diagrams and terminology that are the currency of modern mathematical communication at all levels, while trying at the same time to remain faithful to the original content; very occasionally and mainly as context, there will even be evocations of more modern concepts. Such an approach also helps to counter the temptation of looking at Indian mathematics with its non-Greek pedigree as something exotic and alien, done by people with strange ways of thinking, to be understood after strenuously mastering strange and difficult languages. The hope is that the reader will see at the end that the mathematics of India is the same mathematics that he or she is used to – and that applies also to Indian readers. It goes without saying that some of the mathematics so conveyed will be elementary, school and college mathematics of present times. But there will also be topics which, inherently and in their treatment, we will consider today to have been sophisticated and modern in spirit. Examples would include Aryabhata's introduction of functional differences, Brahmagupta's method of generating new solutions of the quadratic Diophantine equation from known ones, the quasi-axiomatic treatment in *Yuktibhāṣā* of the set of positive integers, possibly as preparation for inductive proofs, and many others – modernity is not necessarily an increasing function of time. An abstract notation and the mindset that goes with it are an advantage in assigning such advances their proper place in the evolutionary chain of mathematics as a whole. At the same time, and equally usefully, the detachment that results from a degree of abstraction will let us see more clearly the difficulty a verbally expressed mathematical culture faces in extending and generalising knowledge acquired in a particular context.

But translating mathematical verse and prose into an abstract-symbolic language also has its risks. It is all too easy to read more into a piece of text than was actually there, e.g., to see in the equation representing a verbal passage all that that equation conveys to the prepared mind of today. There is also the opposite risk, that of devaluing the originality and power of an idea or method because everything that a modern mathematical sensibility expects to read into its symbolic expression is not to be found in its first formulation. Examples of both kinds of misreading are easy to come by in the literature, as we shall have occasion to note. A conscious effort to keep out the biases of modernity is nevertheless worth making, even if destined to be not wholly successful.

This is not meant to be a scholarly book. It began life as the notes for a course given to university students of science and its primary goal is to inform and instruct those who, like my students, wish to acquire a general picture of the mathematical culture of India that does not sacrifice essential pedagogic details; all mathematical statements made are proved and the proofs are those in the texts, occasionally streamlined, with notice given, to satisfy the modern norms of mathematical writing. My aim of keeping it reasonably self-contained is also reflected in the way I have selected and organised the bibliographic material. It is by no means exhaustive. There are lists at the end of the book of the indispensable original texts (including translations into English when available) and of secondary sources that I have found most useful. References that relate to specific points are relegated to (infrequent) running footnotes.

0.4 Sanskrit and its Syllabary

Though, in the interests of a general readership, I have avoided quotations in the original Sanskrit, it is difficult to keep out Sanskrit words and phrases altogether, and for more than one reason. There is first the need to convince that translations are verifiably accurate. When more than one way of rendering a particular term suggests itself, or has been suggested in the literature, I have felt it useful to accompany my chosen reading by the original expression. That is a small price to pay for keeping away from inelegancies such as the literal 'seed calculation' (*bījagaṇita*) for algebra or 'debt amount' (*ṛṇasaṃkhyā*) for a negative number. Students of Sanskrit know that it is a precise language and that possible ambiguities can often be resolved by a rigorous reading of the exact terminology employed. Indeed, problems of great mathematical and historical interest which revolve around the phraseology used, and can be resolved by attending to the rules leading to that phraseology, are not unknown.

At the same time, there are also places where different words are used to connote the same concept or, conversely and more problematically, one word may stand for distinct though related concepts. Such deviations from a one-one correspondence between mathematical objects and operations on the one hand and terminology on the other is to be expected in writing that is entirely in narrative form in a natural language. But it can be confusing. The problem afflicts Aryabhata – who's *sūtra* format has little room for precisely constructed names and their explanations – as much as it does the work of the Nila school where terminological inventiveness has a hard time keeping up with the proliferation of innovative ideas and methods.

Related to these is the recourse to certain formulaic word combinations which are designed to convey much more than their literal meaning. The sense to be attributed to them is decided either by convention or the mathematical context in which they occur. Perhaps the most common is *trairāśikam* ('of three numbers') generally translated as the 'rule of three', the rule which determines any number out of four, which are in equal proportion pairwise, from the other three. Even more rich in content is the Nila school's *jīveparasparanyāyam*, 'the principle of adjacent chords', which stands for the addition formula for the sine function *and* Madhava's derivation of it. In neither of these cases is the full meaning of the formula in doubt but that is not always true.

These are all good reasons for the distracting presence of the occasional Sanskrit phrase. Mainly they serve as guideposts to help us stay on the straight and narrow path of fidelity to the texts, especially in a book such as this that is twice removed from the original material, translation into English followed by a rephrasing in modern mathematical terms. There is however one area into which it is virtually impossible to venture without getting involved in the linguistic and grammatical subtleties of Sanskrit – where the mathematics is in the language itself – namely the development of the decimal place-value system of numeration. The precise quantification or 'measurement' of a general number (the cardinality of an arbitrary finite set) by means of 10 as base in early Vedic times came about orally, not through symbols for the numerals up to 9 and their relative positions in a written representation, but through a choice of names for these numerals as well as for the 'places', the powers of 10. To get the decimally determined name for any number, the primary number names had to be combined following the grammatical rules of Vedic (and, by and large, later) Sanskrit, the rules of nominal composition. What is a simple and easily grasped matter of ordering in a written positional notation is thus subsumed in the elaborate rules governing the composition of words and syllables. I have tried to keep grammatical excursions to the essential minimum, but a feeling for these rules is of great help in recognising the mathematical role they played in Indian decimal enumeration. Indeed, the early symbiosis between numbers and words remained a potent influence on both grammar and arithmetic for a long time.

The Sanskrit words and phrases that are present in this book are written in Roman script, making use of diacritical marks according to standard (IAST) practice. Guides to the way the marks determine pronunciation can be found easily (on many Internet sites for example). But since the syllabary they encode follows certain principles of organisation which have had at least two mathematical manifestations – Pingala (Piṅgala)'s combinatorial approach to prosody and a particular syllabic transcription of numbers, the so-called *kaṭapayādi* enumeration (we will come back to them in due course) – I use this occasion to describe its organisation very briefly. In any case, it is very much a part of the history of knowledge in our first phase; there are in fact scholars who have expressed the view that the science of language in all its aspects is the first science of India.

The currently accepted picture of the history of the Vedic people is that sections of them moved east and south over several centuries, eventually inhabiting most of the Gangetic plain. By about the 5th century BCE, they had a strong presence as far east as present day Bihar and it is to this time and place that the fixing of the final form of the Sanskrit syllabary is credited. It has remained virtually unchanged to the present, and has in fact become, with very minor changes, the syllabary of the majority of the main modern Indian languages (though there are marked differences between them in their written form).

We have, first, the vowels:

a ā i ī u ū e ai o au

In the pronunciation, the unbarred *a*, *i* and *u* have the same duration, called 'short', while their barred variants, called 'long', have twice that duration. The diphthongs *ai* and *au* are long, consistent with how we articulate them, but the surprise is that *e* and *o* are also long in Sanskrit (no bar on them since the short variants never occur). I add for future reference that in Malayalam, the language of *Yuktibhāṣā*, *e* and *o* do occur in both short and long versions like the other vowels, the long being distinguished, in that context, by an overbar.

To these have to be added the 'semivowels' *ṛ* and *ḷ* which come after *ū*, as well as the 'seminasal' *anusvāra* (*ṃ*) and the 'semi-aspirate' *visarga* (*ḥ*) at the end of the list.

Apart from being syllables in their own right, the vowels make possible the vocalisation of the consonants which also therefore come in two durations. The main body of the consonants (conventionally vocalised with the short *a*, any other vowel will do as well) is arranged as a grid, the 'consonant square' of five groups of five syllables each:

ka	*kha*	*ga*	*gha*	*ṅa*
ca	*cha*	*ja*	*jha*	*ña*
ṭa	*ṭha*	*ḍa*	*ḍha*	*ṇa*
ta	*tha*	*da*	*dha*	*na*
pa	*pha*	*ba*	*bha*	*ma*

The syllables are to be read in the normal sequence, first horizontally and then vertically. Apart from these twentyfive, there are nine sundry consonants which do not fit into the grid and so are appended at the end:

ya *ra* *la* *va* *śa* *ṣa* *sa* *ha* (plus *ḷa* in Malayalam).

The point about the grid is that as we proceed down from the top row, the vocalising point for the production of the corresponding sound moves progressively from the farthest part of the palate towards the front, to the teeth and then the lips. The syllables with the underdot (the middle row as well as *ḷa*) are all retroflex, for example. And as we proceed from the extreme left column to the right, the primary sound (*ka*, *ca*, etc.) acquires a modulation depending on the simultaneous use of the breath or on minor changes in the sound-producing action. Thus the second and the fourth column are the aspirates and the fifth the nasals.

A compound consonant (generally of two 'pure' consonants, occasionally of three, as for example in *matsya, śāstra*) is one syllable, pronounced with only the final consonant (*ya, ra* in the examples) having a vowel value, the preceding ones being employed without vowel value or duration. This is not specific to Sanskrit (though there are Indian languages which do not follow this practice). It is also specified that the syllable preceding a compound consonant is to be pronounced long even when it is, in isolation, short. This only codifies something we know from actual speech: the duration of each syllable in the pronunciations of *matsya* and *māsa* for instance is the same even though one word has the short *ma* and the other the long *mā*.

In the interest of readability, I have avoided the use of diacritical marks on the names of people and places, but have indicated the correct pronunciation in brackets on their first occurrence when it was felt to be useful. The names of individual texts are generally given in full diacritical glory since often their meanings do matter.