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Erdélyi–Kober Fractional Calculus

From a Statistical Perspective, Inspired by Solar Neutrino Physics

Springer

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Preface

This monograph deals with Erdélyi-Kober fractional integrals and fractional derivatives from a statistical perspective, inspired by solar neutrino physics. The application of diffusion entropy analysis to Super-Kamiokande data led to ideas to consider generalizations of entropy (entropic pathway) and diffusion (anomalous diffusion). Examplified by Erdélyi-Kober fractional calculus it is shown that the statistical density of a product of two statistically independently distributed real scalar positive random variables or real positive definite matrix-variate random variables or complex Hermitian positive definite matrix-variate random variables is a constant multiple of the Erdélyi-Kober fractional integral of the second kind of order α and parameter γ, when the density of one of the random variables is arbitrary and the density of the other random variable is a type-1 beta density with the parameters $(\gamma+1,\alpha)$ in the real scalar case, $(\gamma+\frac{p+1}{2},\alpha)$ in the real $p\times p$ matrix-variate case, and $(\gamma+p,\alpha)$ in the complex $p\times p$ matrix-variate case. If $x_1>0, x_2>0$ are the real scalar random variables, $X_1>O, X_2>O$ are real $p\times p$ matrix-variate random variables, $\tilde{X}_1>O, \tilde{X}_2>O$ are the complex $p\times p$ matrix-variate random variables, then the product is $u_2=x_1x_2$ (real scalar case), $U_2=X_2^{\frac{1}{2}}X_1X_2^{\frac{1}{2}}$ (real matrix-variate case), and $\tilde{U}_2=\tilde{X}_2^{\frac{1}{2}}\tilde{X}_1\tilde{X}_2^{\frac{1}{2}}$ (complex matrix-variate case). Then the density of $u_2, U_2, and \tilde{U}_2$, denoted by $g_2(u_2), g_2(U_2), and \tilde{g}_2(\tilde{U}_2)$, respectively, is a constant multiple of the Erdélyi-Kober fractional integral of the second kind. It is shown that the density of the ratio $u_1=\frac{x_2}{x_1}, U_1=X_2^{\frac{1}{2}}X_1^{-1}X_2^{\frac{1}{2}}, and \tilde{U}_1=\tilde{X}_2^{\frac{1}{2}}\tilde{X}_1^{-1}\tilde{X}_2^{\frac{1}{2}}$, in the real scalar case, in the real $p\times p$ matrix-variate case, and in the complex $p\times p$ matrix-variate case, respectively, is a constant multiple of the Erdélyi-Kober fractional integral of the first kind of order α and parameter γ when one density is arbitrary and the other density is a type-1 beta density with parameters (γ,α) in all the cases. When the functions are not densities, then it is shown that the second kind and first kind Erdélyi-Kober fractional integrals are the Mellin convolution of a product and ratio, respectively, in the scalar case and M-convolution of a product and ratio in the matrix-variate case. General definitions of first kind and second kind Erdélyi-Kober fractional integral operators are established, from where all the various fractional integrals introduced in the literature are available as special cases. These ideas are extended to the real and complex matrix-variate cases. From these fractional integral operators, fractional differential operators are derived, both in the Riemann-Liouville and Caputo senses.

Chapter 1 provides a brief overview on solar neutrino detection and its background in terms of statistical mechanics and neutrino physics. Results of the diffusion entropy analysis of solar neutrino data collected by Super-Kamiokande are provided and discussed in terms of a prospective fractional diffusion model that leads to a diffusion equation in terms of Erdélyi-Kober operators. This result is the basis for the development of Erdélyi-Kober fractional calculus in the following chapters from a statistical perspective.

Chapter 2 covers Erdélyi-Kober fractional integrals in the real scalar variable case. A general notation is introduced to cover all fractional integrals and fractional derivatives. It is shown that all the fractional integrals available in the literature can be obtained as special cases from the general definition given here in terms of statistical densities of product and ratios, Mellin convolutions of products and ratios, or M-convolutions of products and ratios.

Chapter 3 deals with Erdélyi-Kober fractional integrals in the real matrix-variate case. Connections to statistical densities of product and ratio of matrix-variate random variables are also established here.

Chapter 4 introduces Erdélyi-Kober fractional integrals for the real multivariate case. Multivariate means a collection of real scalar variables and real-valued functions of these variables.

Chapter 5 generalizes these scalar variable results to real matrix-variate cases.

Chapter 6 starts with the discussion of Erdélyi-Kober fractional integrals in the complex domain. The necessary tools for handling real-valued scalar functions of matrix argument, when the argument matrix is Hermitian positive definite, are developed in this chapter. Connections to complex matrix-variate statistical distributions are also established. The basic idea in all these developments is statistical densities of products and ratios and their connection to Erdélyi-Kober fractional integrals.

In Chap. 7, differential operators, operating on real and complex matrices, are developed. With the help of these differential operators, fractional derivatives in the real and complex matrix-variate cases are derived from the corresponding fractional integrals in Chaps. 3 and 6. Here, the operators introduced can work only on certain types of functions of real and complex matrix argument and hence not universal. This area is open to come up with universal differential operators operating on real-valued functions of real and complex matrix argument.

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20 February 2018

Contents

Acronyms

dX, $d\tilde{X}$	: wedge product of differential/Sect. 2.2
$X > O$, $\tilde{X} > O$	: positive definite matrices; real case, complex case/Sect. 2.2
$\int_A^B f(X)dX$	: integral over matrices/Sect. 2.2
$\det(X)$, $\|X\|$	: determinant of X, real case/Sect. 2.2
$\|\det(\tilde{X})\|$	: absolute value of the determinant of $\tilde{X}$/Sect. 2.2
$D_{1,(a,x)}^{-\alpha} f$	: Riemann-Liouville fractional integral of the first kind/ Sect. 2.2
$D_{2,(x,b)}^{-\alpha} f$	: Riemann-Liouvile fractional integral of the second kind/ Sect. 2.2
$W_{1,x}^{-\alpha} f$	: Weyl fractional integral of the first kind/Sect. 2.2
$W_{2,x}^{-\alpha} f$	: Weyl fractional integral of the second kind/Sect. 2.2
$K_{1,x,\zeta}^{-\alpha} f$	: Erdélyi-Kober fractional integral of the first kind/Sect. 2.2
$K_{2,x,\zeta}^{-\alpha} f$	: Erdélyi-Kober fractional integral of the second kind/Sect. 2.2
$S_{1,x,\beta,\gamma}^{-\alpha} f$	: Saigo fractional integral of the first kind/Sect. 2.2
$S_{2,x,\beta,\gamma}^{-\alpha} f$	: Saigo fractional integral of the second kind/Sect. 2.2
Replace $-\alpha$ by $+\alpha$	for the corresponding fractional derivatives; replace small x by U_1, U_2 for fractional integrals of the first and second kind in the real matrix-variate cases; replace U_1, U_2 by $\tilde{U}_1, \tilde{U}_2$ for the corresponding fractional integrals in the complex matrix-variate cases
$g_1(u_1)$, $g_{1j}(u_1)$	: density of the ratio or Mellin convolution of a ratio/Sect. 2.3
$E(\cdot)$	: expected value of $(\cdot)$/Sect. 2.3
$g_2(u_2)$, $g_{2j}(u_2)$	: density of a product or Mellin convolution of a product/ Sect. 2.9
$A^{\frac{1}{2}}$	: square root of a positive definite matrix A/Sect. 3.1
$\Gamma_p(\alpha)$	: real matrix-variate gamma function/Sect. 3.1
$B_p(\alpha, \beta)$	: real matrix-variate beta function/Sect. 3.1
$g_2(U_2)$	: density of a product or M-convolution of a product, real matrix-variate case/Sect. 3.2

$(a)_K$	: generalized Pochhammer symbol/Sect. 3.5
${}_mF_n$	: hypergeometric series/Sect. 3.5
$C_K(Z)$	: zonal polynomial, real case/Sect. 3.5
$\Gamma_p(\alpha, K)$	: gamma on partitions, real case/Sect. 3.5
$K_{2,u_j,\zeta_j,j=1,\ldots,k}^{-\alpha_j,j=1,\ldots,k}f$	: multivariate second kind Erdélyi-Kober fractional integral/Sect. 4.1
$K_{1,u_j,\zeta_j,j=1,\ldots,k}^{-\alpha_j,j=1,\ldots,k}f$	: multivariate first kind Erdélyi-Kober fractional integral/Sect. 4.4
$\tilde{\Gamma}_p(\alpha)$	: complex matrix-variate gamma function/Sect. 6.1
$\tilde{B}_p(\alpha, \beta)$	: complex matrix-variate beta function/Sect. 6.1
$(a)_K$	: generalized Pochhammr symbol, complex case/Sect. 6.4.3
$\tilde{\Gamma}_p(\alpha, K)$	: gamma on partitions, complex case/Sect. 6.4.3
${}_m\tilde{F}_n$	: hypergeometric series, complex case/Sect. 6.4.3
$\tilde{C}_K(\tilde{Z})$	: zonal polynomial, complex case/Sect. 6.4.3
$\tilde{g}_2(\tilde{U}_2)$	: M-convolution of a product, complex case/Sect. 6.4.3
$\tilde{g}_1(\tilde{U}_1)$	: M-convolution of a ratio, complex case/Sect. 6.5
D_{U_-}, D_{U_+}	: determinant of differential operators, matrix-variate case/Sect. 7.2