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# Algebraic Topology

## A Primer

Second Edition

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# Preface

Algebraic Topology is an important branch of topology having several connections with many areas of modern mathematics. Its growth and influence, particularly since the early forties of the twentieth century, has been remarkably high. Presently, it is being taught in many universities at the M.A./M.Sc. or beginning Ph.D. level as a compulsory or as an elective course. It is best suited for those who have already had an introductory course in topology as well as in algebra. There are several excellent books, starting with the first monograph 'Foundations of Algebraic Topology' by S. Eilenberg and N.E. Steenrod, which can be prescribed as a textbook for a first course on algebraic topology by making proper selections. However, there is no general agreement on what should be the 'first course' in this subject. Experience suggests that a comprehensive coverage of the topology of simplicial complexes, simplicial homology of polyhedra, fundamental groups, covering spaces and some of their classical applications like invariance of dimension of Euclidean spaces, Brouwer's Fixed Point Theorem, etc. are the essential minimum which must find a place in a beginning course on algebraic topology. Having learnt these basic concepts and their powerful techniques, one can then go on in any direction of the subject at an advanced level depending on one's interest and requirement.

This book is designed to serve as a textbook for a first course in algebraic topology described above. In order to lay down the real foundation of algebraic topology, we have also included a brief introduction to singular homology and cohomology. Approach to the contents has been dictated by several advanced courses taught by the author on different topics of algebraic topology at many universities. It is necessary to mention that the subject appears a bit abstract to begin with, but after a while it presents concrete topological as well as geometrical results of great insight and depth. The beginning student is expected to have some patience before appreciating the depth of these results. Maximum care has been taken to emphasize the subtle points of the concepts and results in a lucid manner. Examples are given at every stage to illustrate and bring out the underlying concepts and the results. It is hoped that the detailed explanations will help the student to grasp the results correctly and to have a sound understanding of the subject with confidence. The prerequisites for the study of this book are very little, and these have been briefly discussed in Chapter 1 and the Appendix to make it self-contained. These are included to maintain continuity between what the student has already learnt and what he is going to learn. It may be either quickly reviewed or even skipped.

We introduce and study the fundamental groups and its properties in Chapter 2. Starting with the concept of pointed spaces we show that the fundamental groups are topological invariants of path-connected spaces. After computing the fundamental group of the circle, we show how it can be used to compute fundamental groups of other spaces by geometric methods. In Chapter 3, we explain the topology of simplicial complexes, introduce the notion of barycentric subdivision and then prove the simplicial approximation theorem. In Chapter 4, we introduce the first classical homology theory, viz., the simplicial homology of a simplicial complex and then proceed to define the simplicial homology of a compact polyhedron. We provide detailed discussion of how a continuous map between compact polyhedra induces a homomorphism in their simplicial homology, and then prove the topological invariance of these groups. A few classical applications of the techniques of homology groups include the proof of the fact that the Euclidean spaces of different dimensions are not homeomorphic, Brouwer's Fixed Point Theorem, Lefschetz Fixed Point Theorem and the Borsuk-Ulam Theorem, etc. In Chapter 5, we study the theory of covering projections and its relation with fundamental groups. Important results on lifting of a map, classification of covering projections and the universal coverings are discussed at length.

The final topic dealing with the singular homology and cohomology has been discussed in Chapter 6. We have presented the important properties of singular homology including proofs of the homotopy axiom and the excision axiom. All of this is done in a way so that singular homology gets established as a 'homology theory' on the category of all topological pairs in the sense of Eilenberg-Steenrod. Then we go on to formally explain the definitions of an abstract homology as well as an abstract cohomology theory. Apart from the singular homology, we show that simplicial homology is also a homology theory on the category of all compact polyhedral pairs. Related topics such as homology and cohomology with coefficients, the Universal Coefficient Theorem, the Künneth Formula, the Mayer-Vietoris Sequence and the cohomology algebra of a space etc., have been discussed and adequately illustrated by examples. The student can test his understanding of the subject by working out the exercises given in every chapter. All of the basic material covered in this book should make the subject fascinating for beginners and should enable them to pursue advanced courses in any branch of algebraic topology.

Except for a few remarks, we have refrained from giving a detailed history of the subject which is, of course, always very enlightening and interesting.

However, we feel that it can be appreciated adequately when the student has learnt the subject at a more advanced level than what is presented here. The material presented here is contained in most of the books on algebraic topology. My contribution is basically in organizing and presenting it afresh. Definitions, propositions, corollaries etc. are numbered by 3 digits, i.e., Theorem 4.5.1 means Theorem 1 of Section 5 of Chapter 4. A suggested guideline for teaching a two-semester (or one academic year) course from this book is as follows: in the first semester one can cover Fundamental Group (Sections 2.1 to 2.6), Simplicial Complexes (Sections 3.1 to 3.4), Simplicial Homology (Sections 4.1 to 4.8) and Covering Projection (Sections 5.1 to 5.6). The chapter on Covering Projection can be taught immediately after the chapter on Fundamental Group. Proofs of some of the technical theorems given in Chapters 4 and 5 can be omitted. The remaining section of Chapter 4 on Simplicial Homology, viz. Section 4.9 and the whole of Chapter 6 on Singular Homology can be easily covered in the second semester. It is hoped that the material prescribed for the first semester is quite adequate for acquainting the student with the basic concepts of algebraic topology. The material for the second semester, on the other hand, will allow him to take a deeper plunge in any of the advanced topics of modern algebraic topology.

The author would like to express his thanks to many of his friends and colleagues, specially to G.A. Swarup, A.R. Shastri, Ravi Kulkarni and K. Varadarajan, all of whom have read the manuscript and made important suggestions for improvement of the book. The author appreciates the facilities made available to him by the Department of Mathematics, University of Arkansas, where the first draft of the whole book was completed. Rajendra Bhatia, the Managing Editor of the series 'Texts and Readings in Mathematics' of the Hindustan Book Agency is thanked for his keen interest in the editorial work. I thank my wife Prema Tripathi for her insistence that the job of completing the book should get preference over administrative duties. Finally, J.K. Maitra, one of my students and a colleague, deserves special thanks for his constant assistance in typesetting the book on LaTeX.

Satya Deo

July 1, 2003 Jabalpur

## Preface to the Second Edition

In this edition of 'Algebraic Topology, A Primer', some necessary changes have been made in Chapters 3, 4 and 6. In the original edition, a simplicial complex was assumed, for simplicity, to be finite, which meant that all the polyhedra were assumed to be compact. This was indeed a sweeping restriction, especially when one is defining singular homology on the category of all topological pairs. Therefore, Chapter 3, has now been expanded by including another section on general simplicial complexes. As a consequence of this, basic results on the topology of a simplicial complex have also been included. Then, Chapter 4, on simplicial homology has been expanded to include simplicial homology of an arbitrary polyhedral pair. This allows us to state the important fact later in Chapter 6 that the simplicial homology is a homology theory on the category of all polyhedral pairs and their maps in the sense of Eilenberg-Steenrod.

Earlier, the Chapter 6 on singular homology was really somewhat sketchy. It has now been expanded with more details. In order that the book remains only elementary and within size, we have not included the proof of theorems such as the Acyclic Model Theorem, Eilenberg-Zilber Theorem nor the properties of the cup products in singular cohomology. However, the classical applications like the Jordan-Brouwer Separation Theorem, Jordan Curve Theorem and the Invariance of Domain Theorem, etc. have now been included in reasonable detail.

Many colleagues and students have pointed out typos and even a few mathematical errors in the original edition. The author is thankful to all of those, especially to H.K. Mukherjee, Krishnendu Gangopadhyay and Snigdha Bharati Choudhury, who read the book carefully and made many comments. All the corrections have now been made to the best of my knowledge. The numbering of definitions, theorems, corollaries, propositions, lemmas, examples and remarks have now been made in one sequence, i.e., Proposition 3.2 will follow Lemma 3.1 and Theorem 6.4 will follow Proposition 6.3 etc. Some exercises have been changed and the remaining ones have been properly ordered. The author is thankful to all those who followed this book as a text for a course on the subject and offered their valuable suggestions for improving the book. The 'Primer Character' of the book, however, has not been changed. My student Dr V.V. Awasthi of VNIT, Nagpur deserves thanks for helping me in the typesetting of the text. Finally, the author expresses his sincere thanks to the authorities of the Harish-Chandra Research Institute, Allahabad for the wonderful facilities and to the National Academy of Sciences, India for the Platinum Jubilee Senior Scientist Fellowship.

Satya Deo
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May 29, 2017

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SATYA DEO is senior scientist at the National Academy of Sciences, India, and honorary professor at Harish-Chandra Research Institute, Allahabad, India. He was awarded the Distinguished Service Award by the Mathematical Association of India for outstanding contributions to the cause of teaching and research in 2006 and the Indian Science Congress Gold Medal by the Prime Minister of India, in 2010. He has guided 17 PhD students and published over 70 papers in reputed international journals and proceedings. His area of specialization includes algebraic and differential topology, topological and differentiable group actions, homology-cohomology theories, cohomologicaldimesion theory, burnside rings, hopfian and cohopfian rings, spline modules and topological combinatorics.