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Shun-Qing Shen

# Topological Insulators

Dirac Equation in Condensed Matter

Second Edition

 Springer

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# Preface to the Second Edition

The field of topological materials has been developing very rapidly since the first publication of this book. The experimental observation of the quantum anomalous Hall effect in magnetically doped topological insulator thin films and the discovery of the topological semimetals are two outstanding examples. Measurement of Majorana fermions has been reported by several experimental groups. Besides topological band structures have gone beyond quantum materials and are found in materials such as the photonic crystals, metamaterials, and even mechanic systems. Thus, the topological quantum phenomena and related materials now have become an important and inalienable part in condensed matter physics and material sciences.

In this edition, Chap. 11 on topological semimetals and several sections on experimental observation of the quantum anomalous Hall effect, the quantum spin Hall effect, and Majorana fermions are added. Also the first and last chapters have been updated. Here I would like to thank Drs. Hai-Zhou Lu, Song-Bo Zhang, and Jianhui Zhou for their contributions and helpful discussions on Weyl semimetals.

Hong Kong, China  
March 2017

Shun-Qing Shen

# Preface to the First Edition

Recent years, we have seen rapid emergence of topological insulators and superconductors. The field is an important advance of the well-developed band theory in solids since its birth in 1920s. The band theory or Fermi liquid theory and Landau's theory of spontaneously broken symmetry are two themes for most collective phenomena in many-body systems, such as semiconductors and superconductors. Discovery of the integer and fractional quantum Hall effects in 1980s opens a new window to explore the mystery of condensed matters: Topological order has to be introduced to characterize a large class of quantum phenomena. Topological insulator is a triumph of topological order in condensed matter physics.

The book grew out of a series of lectures I delivered in an international school on "Topology in Quantum Matter" at Bangalore, India, in July 2011. The aim of this book is to provide an introduction for a large family of topological insulators and superconductors based on the solutions of the Dirac equation. I believe that the Dirac equation is a key to the door of topological insulators. It is a line that could thread all relevant topological phases from one to three dimensions, and from insulators to superconductors or superfluids. This idea actually defines the scope of this book on topological insulators. For this reason, a lot of topics in topological insulators are actually not covered in this book, for example, the interacting systems and topological field theory. Also I have no ambition to review rapid developments of the whole field and consequently no intention to introduce all topics in this introductory book.

I would like to express my gratitude to my current and former group members, and various parts of the manuscript benefited from the contributions of Rui-Lin Chu, Huai-Ming Guo, Jian Li, Hai-Zhou Lu, Jie Lu, Wen-Yu Shan, Yan-Yang Zhang, An Zhao, Yuan-Yuan Zhao, Rui Yu, and Bin Zhou. Especially, I would like to thank Hai-Zhou Lu for critical reading the manuscript and replotting all figures. I benefited from numerous discussions and collaborations with Qian Niu,

Jainendra K. Jain, Jun-Ren Shi, Zhong Fang, and Xin Wang on the relevant topics. I am grateful for the support and suggestions from Lu Yu while writing this book. Some of the results in this book were obtained in my research projects funded by Research Grants Council of Hong Kong.

Hong Kong, China  
June 2012

Shun-Qing Shen

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