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Multivariate Wavelet Frames

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Preface

Wavelet theory lies at the intersection of pure and computational mathematics, as well as of audio and graphics signal processing, including compression and transmission of information. Wavelet bases have several advantages compared with other bases used as approximation tools. One of them is the so-called time-frequency localization property: Wavelet basis functions as well as their Fourier transformations rapidly decay at infinity. Through this property, in the decomposition into the basis of signals, frequency characteristics of which vary according to time or space, many expansion coefficients with unnecessary at this spatial or temporal area harmonics are small and can be discarded, thereby providing data compression.

Wavelet frames (framelets) are actively used for the same purposes. Moreover, they are very efficient in the image recovery from incomplete observed data, including the tasks of inpainting and image/video enhancement. In the recovery of missing data from incomplete and/or damaged and noisy samples, application of wavelet methods based on frames is more advanced due to the redundancy of frame systems.

Multivariate wavelet systems with matrix dilations (so-called nonseparable wavelets) have been increasingly used for digital processing of multidimensional signals such as images, videos, tomography, and seismic and other signals. Nonseparable wavelets turn out to be more natural in signal processing because multidimensional signals are usually nonseparable. Nonseparable filter banks have better characteristics than their separable counterparts (which consist of products of 1-D filter banks along each dimension). The number of degrees of freedom is also much bigger for nonseparable filter banks. In tomography, the 2-D separable wavelets impose a rectangular tiling of the frequency plane, which is not well suited to the radial band-limited assumption of the image. The application of nonseparable multiresolution tomography to 2-D wavelets allows us to respect the geometry of the system by tiling the frequency plane in a diamond-shaped fashion that is more suitable to the radial band-limited assumptions. Local tomography using these nonseparable bases shows an improvement in terms of PSNR. Another successful application of nonseparable wavelets was in 3-D rotational angiography.

This book presents a systematic study of the theory and methods for the construction of multivariate wavelet frames with matrix dilation, in particular, orthogonal and biorthogonal bases which are a special case of frames. Construction of multivariate nonseparable wavelet frames, especially bases with desirable properties, is a challenging problem. Though a general scheme of construction is well known, its practical implementation in the multidimensional setting is difficult.

We describe methods for the construction of wavelet frames with a matrix dilation providing an arbitrary approximation order and other important features. We also discuss possible conditions under which a frame constitutes a basis. Applied mathematicians and engineers are especially interested in the construction of compactly supported wavelet systems. We give algorithmic methods for the construction of dual and tight compactly supported wavelet frames. Another important feature is symmetry. Different kinds of symmetry of wavelets are very much desirable in various applications, since they preserve linear-phase properties and also allow symmetric boundary conditions in wavelet algorithms which normally perform better. We discuss how to provide H-symmetry, where H is an arbitrary symmetry group, for wavelet bases and frames. The so-called frame-like wavelet systems and their approximation properties are also studied. The frame-like systems inherit many advantages of frames and can be used in applications instead of frames, although their construction is much simpler. The smoothness of wavelets is also important in engineers' problems. To provide smoothness of a wavelet system, one has to begin with a smooth generating refinable function. We extend the matrix method of computing the regularity of refinable function from the univariate case to multivariate refinement equations with arbitrary dilation matrices. This makes it possible to find the exact values of the Hölder exponent of refinable functions and to make a very refine analysis of their moduli of continuity.

The book will be useful for engineers working in signal processing, who can use algorithmic methods for wavelet construction or extract ready-made examples of masks. It can also be interesting for specialists working in functional analysis, approximations, numerical PDE, and related areas. Most of the material is available for graduate students familiar with the basics of functional and real analysis on the level of a standard university course.

The list of references contains a lot of papers which are related to the topics we discuss. Not all of them are cited, and not all results and ideas of these papers are presented in the book.

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Basic Notation

$\mathbb{N}$ is the set of positive integers, $\mathbb{R}^d$ is the d-dimensional Euclidean space, $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d)$ are its elements (vectors), $\mathbf{0} = (0, \ldots, 0)$, $(x, y) = x_1y_1 + \cdots + x_dy_d$, $|x| = \sqrt{(x, x)}$, $\mathbb{R} = \mathbb{R}^1$, $\mathbb{Z}^d$ is the integer lattice in $\mathbb{R}^d$, $\mathbb{Z} = \mathbb{Z}^1$, $\mathbb{T}^d = [0, 1)^d$ is the d-dimensional torus. For $x, y \in \mathbb{R}^d$, we write $x \geq y$, if $x_j \geq y_j$ for all $j = 1, \ldots, d$ and we write $x > y$, if $x \geq y$ and $x \neq y$; $\mathbb{Z}^d_+ := \{x \in \mathbb{Z}^d : x \geq 0\}$, $\mathbb{Z}_+ = \mathbb{Z}^1_+$.

If $\alpha, \beta \in \mathbb{Z}^d_+$, $a, b \in \mathbb{R}^d$, we set $[\alpha] = \sum\limits_{j=1}^{d} \alpha_j$, $\alpha! = \prod\limits_{j=1}^{d} \alpha_j!$, $\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\alpha!}{\beta!(\alpha! - \beta)}$, $a^b = \prod\limits_{j=1}^{d} a_j^{b_j}$, $D^\alpha f = \frac{\partial^{[\alpha]} f}{\partial^{\alpha_1} x_1 \ldots \partial^{\alpha_d} x_d}$; $\delta_{\alpha\beta}$ is the Kronecker delta, which is equal to 1 for $\alpha = \beta$; otherwise, it is equal to 0.

If $A, B \subset \mathbb{R}^d$, then $A \pm B = \{x = a \pm b : \ a \in A, b \in B\}$. By ConvA we denote the convex hull of the set A.

$\mathbb{C}^d$ is the d-dimensional complex Euclidean space with the inner product given by $\langle x, y \rangle = \sum_{j=1}^{d} x_j \overline{y_j}$ for $x = (x_1, \ldots, x_d) \in \mathbb{C}^d$ and $y = (y_1, \ldots, y_d) \in \mathbb{C}^d$.

The Lebesgue measure in $\mathbb{R}^d$ is denoted by μ. By a function on $\mathbb{R}^d$ we mean a complex-valued Lebesgue measurable function on $\mathbb{R}^d$. For $1 \leq p \leq \infty$, $L_p(\mathbb{R}^d)$ is the usual Banach space of functions f on $\mathbb{R}^d$ (of the equivalence classes) such that $\|f\|_p := \left(\int_{\mathbb{R}^d} |f(x)|^p dx\right)^{1/p} < \infty$ for $1 \leq p < \infty$ and $\|f\|_\infty :=$ vraisup$_{x \in \mathbb{R}^d} |f(x)|$ for $p = \infty$. For $p = 2$, $L_2(\mathbb{R}^d)$ is a Hilbert space with the inner product given by $\langle f, g \rangle := \int_{\mathbb{R}^d} f(x)\overline{g(x)} dx$.

The support of a function f, i.e., the minimal (with respect to inclusion) closed set such that f is equal to zero almost everywhere on the complement of this set and is denoted by supp f.

χ_e is the characteristic function of a set $e \subset \mathbb{R}^d$; it takes the value 1 at the points $t \in E$ and 0 at all other points.

span $\{f_n, n \in \mathbb{N}\}$ is the set of finite linear combinations of a system $\{f_n\}_{n=1}^{\infty}$ with complex coefficients, span$A$ is the linear span of a set $A \subset \mathbb{R}^d$.

$\mathcal{S}$ is the Schwartz space on $\mathbb{R}^d$, i.e., the space of infinitely differentiable and rapidly decreasing functions on $\mathbb{R}$:

$$\mathcal{S} = \left\{f \in C^{\infty}(\mathbb{R}^d) \mid \|D^{\alpha}f(x)(1+|x|)^k\|_{\infty} < \infty \quad \forall \alpha \in \mathbb{Z}_+^d, \ \forall k \geq 0\right\}.$$

The topology of the space $\mathcal{S}$ is defined as follows:

$$f_j \to 0 \ \Leftrightarrow \|D^{\alpha}f_j(x)(1+|x|)^k\|_{\infty} \to 0 \quad \forall \alpha \in \mathbb{Z}_+^d, \ \forall k \geq 0.$$

$\mathcal{S}'$ is the space of linear continuous functionals on the space $\mathcal{S}$ (the space of tempered distributions).

$\widehat{f}(x) = \int\limits_{\mathbb{R}^d} f(t)e^{-2\pi i(x,t)}\,dt$ is the Fourier transform of a function f from $L_1(\mathbb{R}^d)$; the same notation $\widehat{f}$ is used for the Fourier transform of f which is in $L_2(\mathbb{R}^d)$, or in the space $\mathcal{S}'$ of tempered distributions; $\mathcal{F}$ and $\mathcal{F}^{-1}$ are the operators taking a function to its direct and inverse Fourier transforms, respectively.

For $s \in \mathbb{N}$, $p \geq 1$, the Sobolev space $W_p^s = W_p^s(\mathbb{R}^d)$ consists of all the functions $f \in L_p(\mathbb{R}^d)$ such that

$$\int\limits_{\mathbb{R}^d} |D^{\alpha}f(x)|^p\,dx < \infty \quad \forall \alpha \in \mathbb{Z}^d, [\alpha] \leq s.$$

For $s > 0$, $p = 2$, the Sobolev space $W_2^s = W_2^s(\mathbb{R}^d)$ consists of all the functions $f \in L_2(\mathbb{R}^d)$ such that

$$\int\limits_{\mathbb{R}^d} |\widehat{f}(\xi)|^2(1+|\xi|^2)^s d\xi < \infty.$$

For a continuous function $f : \mathbb{R}^d \to \mathbb{R}$, we denote

$$\omega(f,t) \ = \ \sup_{\|h\| \leq t} \ \|f(\cdot+h) \ - f(\cdot)\|_{C(\mathbb{R}^d)}\ , \qquad t > 0,$$

the modulus of continuity of f. The Hölder exponent is the supremum of numbers $\alpha \geq 0$ such that $\omega(\varphi, t) \leq Ct^{\alpha}$.

$\widehat{f}(k) = \int\limits_{\mathbb{T}^d} f(t)e^{-2\pi i(k,t)}\,dt$ is the k-th Fourier coefficient, $k \in \mathbb{Z}^d$, of a function $f \in L_1(\mathbb{T}^d)$ with respect to the trigonometric system.

If f is a trigonometric polynomial, then $spec(f)$ denotes the spectrum of f, i.e., the set of $k \in \mathbb{Z}^d$ such that $\widehat{f}(k) \neq 0$.

For a countable index set $\mathbb{K}$, $\ell_p(\mathbb{K})$, $1 \le p \le \infty$, is the Banach space of sequences of complex numbers $c = \{c_n\}_{n\in\mathbb{K}}$ with norm $\|c\|_{\ell_p} = \left(\sum_{n\in\mathbb{K}} |c_n|^p\right)^{1/p}$ for $1 \le p < \infty$ or $\|c\|_{\ell_\infty} = \sup_{n\in\mathbb{K}} |c_n|$ for $p = \infty$; $\mathbb{K}$, $\ell_0(\mathbb{K})$ denotes the linear subspace of $\ell_1(\mathbb{K})$ consisting of finite sequences; $\ell_p := \ell_p(\mathbb{N})$.

The inner product of elements f, g of a Hilbert space is denoted by $\langle f, g \rangle$, the norm of element f of a Hilbert space is denoted by $\|f\|$.

For an operator T in a Hilbert space, T^* is the operator adjoint to T; T^{-1} is a norm of the operator.

If A is a $d \times d$ matrix, then $\|A\|$ is its Euclidean operator norm from $\mathbb{R}^d$ to $\mathbb{R}^d$, A^T is its transpose, A^* is its Hermitian conjugate matrix, $\det A$ is the determinant of A; I_d is the identity $d \times d$ matrix.

M denotes a dilation matrix in $\mathbb{R}^d$, i.e., an integer $d \times d$ matrix whose eigenvalues are strictly greater than 1; $m = |\det M|$.

For a function ψ defined on $\mathbb{R}^d$ and a dilation matrix M in $\mathbb{R}^d$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$, $\psi_{jk} = m^{j/2}\psi(M^j \cdot + k)$.

If $\psi^{(v)}$, $v = 1, \dots, r$, are functions defined on $\mathbb{R}^d$ and M is a dilation matrix in $\mathbb{R}^d$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$, then $\{\psi^{(v)}_{jk}\}_{j,k,v} := \{\psi^{(v)}_{jk}, j \in \mathbb{Z}, k \in \mathbb{Z}^d, v = 1, \dots, r\}$