Developments in Mathematics

Volume 44

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Theory of Reproducing Kernels and Applications

Springer

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ISSN 1389-2177 ISSN 2197-795X (electronic)
Developments in Mathematics
ISBN 978-981-10-0529-9 ISBN 978-981-10-0530-5 (eBook)
DOI 10.1007/978-981-10-0530-5

Library of Congress Control Number: 2016951457

Printed on acid-free paper

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Preface

The theory of reproducing kernels started with two papers of 1921 [449] and 1922 [45] which dealt with typical reproducing kernels of Szegö and Bergman, and since then the theory has been developed into a large and deep theory in complex analysis by many mathematicians. However, precisely, reproducing kernels appeared previously during the first decade of the twentieth century by S. Zaremba [497] in his work on boundary value problems for harmonic and biharmonic functions. But he did not develop any further theory for the reproducing property. Furthermore, in fact, we know many concrete reproducing kernels for spaces of polynomials and trigonometric functions from much older days, as we will see in this book. Meanwhile, the general theory of reproducing kernels was established in a complete form by N. Aronszajn [28] in 1950. Furthermore, L. Schwartz [428], who is a Fields medalist and founded distribution theory, developed the general theory remarkably in 1964 with a paper of over 140 pages.

The general theory is certainly beautiful. It seems, however, that for a long time we have overlooked the importance of the general theory of reproducing kernels. We were not able to find an essential reason why the theory is important. Indeed, it was an abstract theory, and from the theory, we were not able to derive any definite results and any essential developments in mathematics. The theory by Schwartz is great; however, its importance remained unnoticed for a long time: It is still ignored.

When we consider linear mappings in the framework of Hilbert spaces, we will encounter in a natural way the concept of reproducing kernels; then the general theory is not restricted to Bergman and Szegö kernels, but the general theory is as important as the concept of Hilbert spaces. It is a fundamental concept and important mathematics. The general theory of reproducing kernels is based on elementary theorems on Hilbert spaces. The theory of Hilbert spaces is the minimum core of functional analysis. However, when the general theory is combined with linear mappings on Hilbert spaces, it will have many relations in various fields, and its fruitful applications will spread over to differential equations, integral equations, generalizations of the Pythagorean theorem, inverse problems, sampling theory, nonlinear transforms in connection with linear mappings, various operators among

Hilbert spaces, and many other broad fields. Furthermore, when we apply the general theory of reproducing kernels to the Tikhonov regularization, it produces approximate solutions for equations on Hilbert spaces which contain bounded linear operators. Looking from the point of view of computer users at numerical solutions, we will see that they are fundamental and have practical applications.

Concrete reproducing kernels like Bergman and Szegö kernels will produce many wide and broad results in complex analysis. They developed some deep theory and lead to profound results in complex analysis containing several complex variables. Meanwhile, the formal general theory by Aronszajn also has favorable connections with various fields like learning theory, support vector machines, stochastic theory, and operator theory on Hilbert spaces.

In this book, we will concentrate on the general theory of reproducing kernels developed by Aronszajn while keeping in mind the theory combined with linear mappings and applications of the general theory to the Tikhonov regularization. We will present many concrete applications from the point of view of numerical solutions for computer use. These topics will be general and fundamental for many mathematical scientists beyond mathematicians as in calculus and linear algebra in an undergraduate course.

One of our strong motivations for writing this book was provided by the historical success of numerical and real inversion formulas of the Laplace transform, which is a famous ill-posed and difficult problem, and, in fact, we will give their mathematical theory and formulas, as clear evidence of the definite power of the theory of reproducing kernels by combining the Tikhonov regularization. For the algorithm based on the theory, Hiroshi Fujiwara made the software and we can use it through his helpful guide.

The web [159] is an open source to his inverse Laplace transform.

For these topics, we will need background materials like integration theory, fundamental Hilbert space theory, the Fourier transform, and the Laplace transform.

In Chap. 1, we will give many concrete reproducing kernels first, and in Chap. 2, we develop the general theory of reproducing kernels with general and broad applications by combining it with linear mappings.

In Chap. 3, we will apply the general and global theory of reproducing kernels to the Tikhonov regularization in a lucid manner. We stand on the point of view of numerical solutions of bounded linear operator equations on Hilbert spaces for computer use in a definite and self-contained way.

Chapter 4 is intended as an introduction to what Hiroshi Fujiwara did. In particular, Fujiwara solved linear simultaneous equations with 6,000 unknowns by means of discretization of a Fredholm integral equation of the second kind. This integral equation of the second kind was derived by the Tikhonov regularization and the reproducing kernel method in the above real inversion formula. At this moment, theoretically we will use all the data of the output—in fact, 6,000 pieces of data. Fujiwara gave solutions in **600 digits** precision with the data of **10 GB** for solutions. This fact had a great impact on the authors. Computer power and its algorithms will improve year by year. Meanwhile, we can practically obtain a finite amount of observation data, and so we expect to obtain solutions in terms of a finite number of data for various forward and inverse problems. Thanks to the power of computers, we will be able to realize more direct and simple algorithms, and so we have included results based on a finite amount of observation data. This method will give a new discretization principle.

Chapter 5 deals with the applications to ordinary differential equations such as fundamental equations $y'' + \alpha y' + \beta y = 0$, where α and β can be general functions. Sometimes, we consider the case when the boundary condition comes into play.

As one main substance of new results, in Chap. 6, we present many concrete results for various fundamental partial differential equations. Here we take up the Poisson equation, the Laplace equation, the heat equation, and the wave equation.

Similarly, in Chap. 7, we deal with integral equations. We will consider typical singular integral equations, convolution equations, convolution integral equations, and integral equations with the mixed Toeplitz and Hankel kernel.

In Chap. 8, we refer to specially hot topics and important materials on reproducing kernels, namely, norm inequalities, convolution inequalities, inversion of an arbitrary matrix, representation of inverse mappings, identification of nonlinear systems, sampling theory, statistical learning theory, and membership problems. This will yield a new method of how to catch analyticity and smoothing properties of functions by computers. Furthermore, we will see basic relationships among eigenfunctions, initial value problems for linear partial differential equations, and reproducing kernels, and we will refer to a new type of general sampling theory with numerical experiments. In the last two subsections, we added new fundamental results on generalized reproducing kernels, generalized delta functions, generalized reproducing kernel Hilbert spaces, and general integral transform theory. In particular, any separable Hilbert space consisting of functions may be viewed as generalized reproducing kernel Hilbert spaces, and the general integral transform theory may be extended to a general framework.

Finally, an appendix is provided. In Sect. A.1, we introduce the theory of Akira Yamada discussing equality problems in nonlinear norm inequalities in reproducing kernel Hilbert spaces; indeed, we may be surprised at his general theory of reproducing kernels. In Sect. A.2, we introduce Yamadafs unified and generalized inequalities for Opialfs inequalities. Similar but different generalizations were independently published by Nguyen Du Vi Nhan, Dinh Thanh Duc, and Vu Kim Tuan, in the same year. In Sect. A.3, we introduce concrete integral representations of implicit functions. We rely upon the implicit function theory guaranteeing the existence of implicit functions. The fundamental result was obtained as a great development of a general abstract theory of reproducing kernels.

Kiryu, Japan — Saburou Saitoh
Hachioji, Japan — Yoshihiro Sawano
November 2015

Acknowledgments

The authors thank the following authors of the textbooks:

1. Alain Berlinet and Christine Thomas-Agnan: *Reproducing Kernel Hilbert Spaces in Probability and Statistics*
2. Baver Okutmustur and Aurelian Gheondea: *Reproducing Kernel Hilbert Spaces: The Basics, Bergman Spaces, and Interpolation Problems* on reproducing kernels that were very instructive for our book

Professor H.G.W. Begehr encouraged the publication of this book.

The three referees gave valuable comments and suggestions to the first draft of this book.

The following mathematicians kindly sent their papers or their text files or kind suggestions for our book publication:

Luis Daniel Abreu, Kaname Amano, Joseph A. Ball, P. L. Butzer, L.P. Castro, Minggen Cui, Hiroshi Fujiwara, Antonio G. Garcia, J. R. Higgins, Hiromichi Itou, M. T. Garayev, Kenji Fukumizu, Tsutomu Matsuura, Yan Mo, J. Morais, Nguyen Du Vi Nhan, Masaharu Nishio, Takeo Ohsawa, Hidemitsu Ogawa, Tao Qian, A. G. Ramm, M. M. Rodrigues, Michio Seto, Fethi Soltani, N. S. Stylianopoulos, Mariko Takagi, Akira Yamada, Masato Yamada, Hiroyuki Yamagishi, Nguyen Minh Tuan, Vu Kim Tuan, Masahiro Yukawa, and Kohtaro Watanabe.

This work of the first author was supported in part by Portuguese funds through the CIDMA (Center for Research and Development in Mathematics and Applications) and the Portuguese Foundation for Science and Technology (FCT), within project PEst-OE/MAT/UI4106/2014.

The first author was also supported in part by the Grant-in-Aid for the Scientific Research (C)(2)(No. 21540111, 24540113) from the Japan Society for the Promotion of Science, and the second author was supported by the Grant-in-Aid for Young Scientists (B) (No.21740104, 24740085) from the Japan Society for the Promotion of Science.

Contents

List of Notation

1. The symbol $\delta_{\cdot_1 \cdot_2}$ denotes the Kronecker delta.
2. Let $K : E \times E \to \mathbb{C}$ be a function on the cross product $E \times E$, where E is a set. Then for $p \in E$, we write $K_p \equiv K(\cdot, p)$.
3. The pointwise product of the functions f and g defined on a set E is denoted by $f \cdot g$, which means $f \cdot g(x) \equiv f(x)g(x)$ for $x \in E$.
4. The tensor product of f and g defined on sets E and F, respectively, is denoted by $f \otimes g$, which means $f \otimes g(x, y) \equiv f(x)g(y)$ for $x \in E$ and $y \in F$.
5. Throughout the whole paper, we denote by C a *positive constant* which is independent of the main parameters, but it may vary from line to line, while $C(\alpha, \beta, \ldots)$ denotes a positive constant depending on the parameters $\alpha, \beta, \ldots$. The symbol $A \lesssim B$ means that $A \le CB$. If $A \lesssim B$ and $B \lesssim A$, then we write $A \sim B$.
6. If E is a subset of $\mathbb{R}^n$, we denote by χ_E the characteristic function of E. For all $a, b \in \mathbb{R}$, let $a \vee b \equiv \max\{a, b\}$ and $a \wedge b \equiv \min\{a, b\}$.
7. For $r > 0$, define $\Delta(r) \equiv \{z \in \mathbb{C} : |z| < r\}$ and $\partial\Delta(r) \equiv \{z \in \mathbb{C} : |z| = r\}$.
8. For an open set $D \subset \mathbb{C}$, the set $\mathscr{O}(D)$ denotes the set of all holomorphic functions on D.
9. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$.
10. Let X be a topological space. Define

$$\mathscr{B}(X) \equiv \{F : X \to \mathbb{C} : F \text{ is bounded continuous}\}. \tag{0.1}$$

11. We use the standard multi-index notation: If we are given points

$$z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \text{ and } \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \{0, 1, 2, \ldots\}^n,$$

then $|\alpha| \equiv \alpha_1 + \cdots + \alpha_n$ and $z^\alpha \equiv \Pi_{i=1}^n z_i^{\alpha_i}$.

12. The space AC$[a, b]$ denotes the space of real-valued absolutely continuous functions on $[a, b]$.
13. Let X and Y be topological spaces such that $X \subset Y$ as a set. If $O \cap X$ is an open set in X for any open set $O \subset Y$, then we write $X \hookrightarrow Y$.
14. Let V be a linear space and E a subset of V. *The span* Span(E) denotes the smallest linear subspace of E containing V.