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Bayesian Theory and Methods with Applications

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Preface

This textbook aimed at the advanced undergraduate and graduate readership is devoted to a systematic account of the fundamentals of the Bayes theory of statistical estimation with applications to the analysis of engineering reliability. Lately, there has been a significant trend toward using the Bayesian approach to develop and analyze problems in different fields of our society. At the same time, the Bayes theory is characterized by an inner logical harmony and simplicity which makes it still more attractive for applied purposes. The application of the bayes theory and methods in the field of reliability gives us the opportunity to save money and time assigned for experiments owing to utilization of relevant prior information instead of the corresponding number of trials.

The subject matter of the book pursues the following double aim:

1. To give an account of the present state of the Bayes theory and methods with emphasis on application.
2. To demonstrate how we can use the Bayes approach for the evaluation of the reliability function in the calculations of reliability which, in practice, covers a great variety of the problems of statistical analysis of engineering reliability.

The distinguishing feature of this monograph is a close unity of fundamental investigation of the main principles of the Bayes theory with clear presentation of its practical applications. The rendering of the fundamentals of the Bayes methodology follows the classical works by Ramsey, Good, Savage, Jefferys, and De Groot, while its present state is represented by the results produced during the last 30 years by a number of scientists from the U.S.A., Canada and countries of Western Europe. The greater part of the monograph is comprised of the presentation of new and original results of the author, the most significant of which are Bayes quasi-parametric estimators, Bayes relative minimax estimators under the conditions of a partial prior information, and the estimators of the working capacity
with an additive error. The Bayes procedures suggested in the monograph are distinguished by a simple way of representation of prior information and use of censored samples that undoubtedly testify to their practical usefulness. The subject methodology presented in this monograph is illustrated with a great number of examples.

Chapter 1 of the monograph plays the role of an introduction and is, in fact, a brief excursion into the history of the Bayes approach. The general principles of the Bayes approach and hierarchical Bayes methodology are discussed in this chapter. Also included are the varieties of subjective probability constructions, as well as an application of the Bayes methodology in the reliability field.

Chapter 2 describes the components of the Bayes approach. In particular, forms of loss functions, choice of the prior probability distribution and the general procedure of reliability estimation are considered.

A systematic description of accepted estimation procedures is given in Chapter 3. The authors demonstrate the process of solving the problems of survival probability estimation from accelerated life tests.

Chapter 4 is devoted to non-parametric Bayes estimation which, in our opinion, is the front line of Bayes theory. Nonparametric Bayes estimators in which the Dirichlet processes are not used are discussed. The authors also consider the nonparametric bayes approach of quantile estimation for increasing failure rate.

A detailed presentation of a new method called “quasi-parametric” is given in Chapter 5. Bayes estimators of a reliability function for a restricted increasing failure rate distribution are studied.

Chapter 6 deals with the class of Bayes estimators of a reliability function under the conditions of partial prior information. The setting of the problem and its general solution that yields a new type of estimator are considered.

Chapter 7 is devoted to empirical Bayes estimators first suggested by Robins. The main results are described briefly. The authors present a new method based on the idea of quasi-parametrical estimation.

Chapters 8–10 are united by common contents that are based on a reliability estimation using functional models of working capacity.

The monograph is addressed (first and foremost) to practicing scientists, though it also deals with a number of theoretical problems. The monograph is a blend of thorough, mathematically-strict presentations of the subject matter and it is easily readable. The monograph can be a useful, authoritative and fundamental source of reference for training
statisticians, scientists and engineers in different fields.

The authors are grateful to several reviewers for their useful suggestions that we have included in the manuscripts: Dr. M. McWaters, Dr. Dennis Koutras, Dr. S. Sambandham, Dr. G. G. Haritonov, Dr. V. B. Chernjavskii, Dr. G. Aryal and Dr. R. Wooten.

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