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Differential Geometry and Mathematical Physics

Part II. Fibre Bundles, Topology and Gauge Fields

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Contents

Introduction

This is the second part of our book on Differential Geometry and Mathematical Physics. It is based on our teaching of these subjects at the University of Leipzig to students of physics and of mathematics and on our research in gauge field theory over many years.

As in Part I, let us start with some historical remarks. The concept of gauge invariance first appeared in the famous papers [660] and [661] of Hermann Weyl from the year 1918.[1] In this work, Weyl extended Einstein's principle of general relativity by postulating that, additionally, the scale of length can vary smoothly from point to point in spacetime. In more detail, Weyl's basic idea was to develop a purely infinitesimal geometry. Behind that concept was his belief that 'a true infinitesimal geometry should, however, recognize only a principle for transferring the magnitude of a vector to an infinitesimally close point …', see page 25 in [660]. In this context, the notion of connection appeared for the first time in the mathematical literature.[2] In a modern geometric language, he was led to a generalization of Riemannian geometry characterized by a pair consisting of a conformal Riemannian structure and a connection in a line bundle over spacetime. Weyl proposed to identify the connection form with the electromagnetic gauge potential and, consequently, its curvature with the electromagnetic field tensor. Thus, he obtained a unification of general relativity with electromagnetism. However, it quickly became clear that this model was not compatible with basic physical principles. It was Einstein who observed that if this theory was correct, then the behaviour of clocks would depend on their history. This is in contradiction with empirical evidence.[3] Although this model did not survive, the gauge principle did though. In 1929 Weyl proposed to apply it to quantum mechanics. He recognized

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[1] In these papers, the term 'gauge invariance' appears in German as 'Maßstab-Invarianz'.

[2] Of course, there were predecessors, notably Christoffel, Ricci and Levi-Civita. The latter had a clear mathematical understanding of parallel transport and of the covariant derivative operator, but up to our knowledge, he did not invent the term 'connection'.

[3] See the postscript by Einstein in [660] and the author's reply. This started a long discussion between Weyl and Einstein. For further reference, see also [604] and [496].

that it is the phase of the Schrödinger wave function which should be gauged, see [663]. In more detail, the idea of Weyl was as follows: since only the absolute value of the wave function has a physical interpretation, the wave function itself may be multiplied by an arbitrary point-dependent phase factor.[4] However, the transformed wave function obviously does not satisfy the Schrödinger equation any more. In order to restore invariance, Weyl proposed to replace the partial derivatives with respect to space and time coordinates occurring in the Schrödinger equation by the covariant derivatives obtained by adding to the partial derivatives the components of the electromagnetic potential. This modified Schrödinger equation is invariant under simultaneous gauge transformations of the wave function and of the electromagnetic potential. This way, the first quantum mechanical model of a U(1)-gauge theory was born.

The combination of this U(1)-gauge principle with the quantum theory of fields led to Quantum Electrodynamics (QED). For an exhaustive historical introduction to that theory we refer to Volume I of [654], see Sect. 1.2. The early contributions to the development of QED date back to the late 1920s and are due to Dirac [152], Weisskopf and Wigner [658], Jordan and Pauli [350], Jordan and Wigner [351] and Heisenberg and Pauli [292]. In the 1930s, QED was studied intensively leading to a further development of the formalism as well as to successful applications. This period culminated in the famous Solvay report by Pauli in 1939, see [505]. Clearly, the biggest puzzle was the emergence of infinities in various kinds of calculations. Amongst a number of approaches to tackle this problem, in the end, the concept of renormalization of the parameters of the theory became the widely accepted strategy. In this spirit, in the late 1940s, Schwinger [579], Tomonaga [629] and Feynman [194] brought QED to its final manifestly relativistic form.[5]

The first non-Abelian gauge theory was proposed by Yang and Mills in 1954, see [685].[6] Their work was based on the idea that the forces between the nucleons were mediated by the exchange of pions and that the interaction was invariant under the isospin group SU(2). In this model, the proton and the neutron form an isospin doublet and the three charged states of the pion form a triplet in the adjoint representation. Yang and Mills postulated the principle of local isotopic gauge invariance. As a consequence, they were led to introduce an SU(2)-gauge potential. They found the field equations of this system, proposed a generalization of the Lorenz gauge fixing condition and made preliminary remarks on the quantum theory of their model. The paper by Yang and Mills dealt with the special gauge group SU(2) only, but from their presentation it was clear how to generalize the

[4]In the group theoretical language, such a transformation is given by a function on spacetime with values in the Abelian group U(1).

[5]For this work, Feynman, Schwinger and Tomonaga received the Nobel Prize in Physics in 1965. Initially, Feynman's diagrammatic technique seemed quite different from the operator-based approach of Schwinger and Tomonaga, but Dyson [171] showed that the two approaches were equivalent.

[6]There was an earlier paper by Klein [378] written in the spirit of Kaluza-Klein theory which already contained a non-Abelian gauge potential.

model to an arbitrary non-Abelian gauge group, see [639] and [236]. It took over ten more years before this seminal paper came into prominence. In 1964 Gell-Mann and Ne'eman [235], [237] proposed SU(3) as the gauge group of strong interactions and in the years 1964–1967 Brout and Englert [106], Higgs [298–300] and Kibble [364] discovered a symmetry-breaking mechanism which gave a mass to some components of the Yang–Mills field. Based on this work and on earlier work by Glashow [247] and others, in the years 1967–1968 Weinberg [654] and Salam [552] unified the electromagnetic and the weak interactions.[7] At the beginning of the 1970s, Gross and Wilczek [264], Politzer [513] and Weinberg [656] created the theory of strong interactions called Quantum Chromodynamics. These theories became the two basic building blocks of the standard model of elementary particle physics.[8]

In the period just described, Weyl's original ingenious understanding that the gauge principle is closely related to the notion of connection did not play any role.[9] The development of the theory of connections evolved in a completely separate way as part of modern geometry and was generally unknown to the physics community. In the beginning of the 1920s, on the basis of his deep expertise in Lie theory and under the influence of Einstein's theory of general relativity and of Klein's Erlangen programme, Élie Cartan started building a general theory of connections with respect to various groups. In contrast to Weyl, who used the absolute differential calculus of Levi-Civita and Ricci, Cartan relied on the calculus of differential forms. In the context of what he called 'generalized spaces',[10] Cartan developed the theory of connections (including torsion) for various types of geometries (Riemannian, Lorentzian, Weylian, affine, conformal, projective and others), see [115–120] and further references in [130] and [568].[11] The next step forward was taken at the beginning of the 1940s by Ehresmann, a student of Cartan, who proposed to use fibre bundles as the natural geometric structure allowing for a global description of a connection, see [174–176] and [410] for further references.[12] As a matter of fact, the very notion of a fibre bundle existed already at that time. It was invented by Seifert [584] as early as in 1932. In the 1930s and 1940s, the study of fibre bundles

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[7]For this work, Glashow, Weinberg and Salam received the Nobel prize in 1979.

[8]For an exhaustive presentation of the history of the standard model see [657].

[9]However, inspired by the work of Einstein, Weyl, Yang, Mills and Utiyama, as early as in 1963, Lubkin [411] made a first step towards the analysis of the geometric content of the gauge concept in terms of connection theory in fibre bundles.

[10]A generalized space in the sense of Cartan is a space of tangent spaces such that two infinitely near tangent spaces are related by an infinitesimal transformation of a given Lie group. Such a structure clearly defines a connection. We note that the tangent space is an abstract notion here, it may not coincide with the space of tangent vectors.

[11]The paper [130] by Chern and Chevalley contains a description of the work of Cartan as a whole. The paper [568] by Scholz gives some interesting insight into the scientific interrelation between Weyl and Cartan.

[12]The paper [410] by Libermann describes the influence of Ehresmann on the development of modern differential geometry in detail.

became a quickly developing field of topology.[13] The main steps were taken by Whitney [665, 666], Hopf and Stiefel [602], Hurewicz and Steenrod [330, 331], Ehresmann and Feldbau (already cited above), Chern[14] [126–129] and Pontryagin [516]. This period culminated in the textbooks on the topology of fibre bundles by Steenrod [599] and on the geometry of connections in fibre bundles by Nomizu [491]. By that time, the theory of fibre bundles was settled as a classical field of geometry and topology. It is beyond the scope of this introduction to describe the further development of this field up until the present time.

The first full description of gauge theory in the language of fibre bundles and connections was presented by Trautman in 1970 [630]. Thereafter, the study of the geometric structure of gauge theories quickly became part of mathematical physics and, within the next decade, quite a number of papers propagating this geometric point of view have been written, see e.g. [161], [173] and [147]. This was related to the fact that, at that time, mathematicians became excited about questions posed by physicists, notably by the question of how to find all self-dual solutions of the Yang–Mills equations. This problem was solved by Atiyah, Drinfeld, Hitchin and Manin [36] using methods of algebraic geometry. In our eyes, this is one of the most fascinating interactions of geometry and physics in the second half of the twentieth century. Via the study of the moduli space of the solutions, it led to deep new insight into the topology of differentiable four-manifolds, see [159]. In the middle of the 1990s, guided by the study of the vacuum structure of $N = 2$ supersymmetric Yang–Mills theory, Seiberg and Witten [582, 583] arrived at a U(1)-gauge model coupled to a spinor field. The investigation of this model gave a new impetus to the study of the topology of differentiable four-manifolds. Within a few months, many of the results obtained via instanton theory were reproved within this new theory and new results, notably in the theory of symplectic manifolds, were obtained. Yet another fruitful interaction of physics and geometry happened in the theory of magnetic monopoles. The three fields of research just mentioned will be discussed in some detail in Chaps. 6 and 7. By the end of the 1970s and the beginning of the 1980s, geometrical and topological methods also started playing a role in quantum gauge theory. This applies, in particular to the study of the Gribov problem and to anomalies. Both of these aspects will be discussed in Chap. 9. Moreover, starting from the beginning of the 1990s, a number of observations, conjectures and results concerning the relevance of the stratified structure of the gauge orbit space for quantum gauge theory appeared. This is one of our fields of research, so we will discuss the structure of the gauge orbit stratification, together with a concept how to implement it on quantum level, in detail in Chaps. 8 and 9.

We continue with a few remarks on the structure and the content of this volume. This volume consists of three building blocks: in the first four chapters we present the geometry and topology of fibre bundles, in Chap. 5 we study the theory of Dirac operators and the remaining four chapters are devoted to gauge theory. In more

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[13] See [434] for a history of the theory of fibre bundles.

[14] See [309] for a detailed description of his mathematical work.

detail, in Chap. 1, we study principal and associated bundles and develop the theory of connections. This includes elementary bundle reduction theory, the theory of holonomy and the theory of invariant connections. In Chap. 2, we study linear connections in the frame bundle of a manifold and their reductions. This leads us to *H*-structures[15] allowing for a unified view on possible geometric structures manifolds may be endowed with. From this perspective, Riemannian geometry occurs as an important special example. In this context, we study compatible connections, the relation of curvature and holonomy and we give an introduction to the theory of symmetric spaces. Moreover, we present elementary Hodge theory and discuss some aspects of 4-dimensional Riemannian manifolds. In Chap. 3, we study the homotopy theory of fibre bundles. We prove the Covering Homotopy Theorem and develop the concept of universal bundles. Using this tool, we prove the fundamental classification theorem for principal bundles in terms of homotopy classes of mappings. We also include a discussion of universal connections. In Chap. 4, we present the basics of the cohomology theory of fibre bundles. We study the cohomology rings of characteristic classes for the classical groups, derive the Whitney Sum Formula and the Splitting Principle and discuss the effect of field restrictions and field extensions. Next, we present the characteristic classes in terms of de Rham cohomology via the Weil homomorphism and discuss the related genera. Finally, we discuss the concept of Postnikov tower and show how it may be used to classify bundles over low-dimensional manifolds. Chap. 5 is devoted to the study of Dirac operators. Given their great importance in gauge theory, we provide the reader with a systematic and quite exhaustive presentation. We start with Clifford algebras, spinor groups and their representations. Next, we discuss spin structures, Dirac bundles and Dirac operators. Since we are going to use the Atiyah–Singer Index Theorem in gauge theory a number of times, we give a full proof of this theorem via the heat kernel method. In the remaining four chapters, we present topics in gauge theory. Clearly, we had to make a choice here, that is, we had to omit a number of interesting topics like, say, topological field theory. In Chap. 6, we study pure gauge theories. We start by deriving the Yang–Mills equations from the variational principle for the Yang–Mills action and show that (anti-)self-dual solutions correspond to absolute minima of the action. We then present a systematic study of instantons: we discuss the BPST-instanton family in detail, present the ADHM-construction and give a partial proof that via that construction one obtains all solutions. In our presentation, we limit our attention to the base space S^4 and to the gauge group $G = SU(2)$. Next, we study the moduli space and outline how it is used for the study of the topology of differentiable 4-manifolds. Finally, we present the classical stability analysis of the Yang–Mills Equation and include a short discussion of non-minimal solutions. In Chap. 7, we include matter fields. We start with the theory of Yang–Mills–Higgs models: we discuss the Higgs mechanism, present a topological classification of static finite-energy configurations and address the problem of constructing asymptotic as

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[15]In the literature, the term *G*-structure is common as well.

well as exact solutions to the Yang–Mills–Higgs equations. In particular, we focus on magnetic monopole solutions including the Bogomolnyi–Prasad–Sommerfield model. Next, we pass to the Seiberg–Witten model. We discuss the basic properties of this model in detail and outline some of the topological consequences. Next, we present the (classical) standard model of elementary particle physics in the geometric language. In the remaining two sections, we give an introduction to the method of dimensional reduction in the context of gauge theories including some of our own results. Chap. 8 is devoted to the study of the gauge orbit stratification. In the first part, we provide the reader with the classical geometrical and topological results on that structure. In the second part, we present our own results on the classification of gauge orbit types in some detail. For clearness of presentation, we limit our attention to the case $G = \mathrm{SU}(n)$. The classification is in terms of characteristic classes (fulfilling a number of algebraic relations) of certain reductions of the principal bundle under consideration. We also show how to derive the natural partial ordering of strata. Finally, in Chap. 9, we come to some elements of quantum gauge theory with the main emphasis on those aspects which are related to the structure of the classical gauge orbit space in one or the other way. In the first part, we present the classical Faddeev–Popov path integral quantization procedure, address the Gribov problem in the language of differential geometry and discuss the classical results of Singer concerning the obstruction against the existence of a global gauge fixing. Next, we discuss anomalies within the geometric setting. In the second part, we present some of our results on non-perturbative quantum gauge theory for (finite) lattice models in the Hamiltonian framework. We construct the quantum model via canonical quantization, derive the field algebra and the observable algebra of the system and discuss the Gauß law. Next, we explain how to include the non-generic gauge orbit strata on the quantum level and discuss their possible physical relevance for a toy model.

We assume that the reader is familiar with the calculus on manifolds as presented in Chaps. 1–4 of Part I and with the theory of Lie groups and Lie group actions as presented in Chaps. 5 and 6 of Part I. For the understanding of Chaps. 3 and 4, basic knowledge in homotopy theory and some elements of algebraic topology are needed. In Chap. 9, we use elements of the theory of C^*-algebras. For the convenience of the reader we have added a number of appendices.