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Noncommutative Geometry and Particle Physics

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Preface

The seeds of this book have been planted in the far east, where I wrote lecture notes for international schools in Tianjin, China in 2007, and in Bangkok, Thailand in 2011. I then realized that an up-to-date text for beginning noncommutative geometers on the applications of this rather new mathematical field to particle physics was missing in the literature.

This made me decide to transform my notes into the form of a book. Besides the given challenge inherent in such a project, this was not made easy because of recent, rapid developments in the field, making it difficult to choose what to include and to decide where to stop in my treatment. The current state of affairs is at least touched upon in the final chapter of this book, where I discuss the latest particle physics models in noncommutative geometry, and compare them to the latest experimental findings. With this, I hope to have provided a path that starts with the basic principles of noncommutative geometry and leads to the forefront of research in noncommutative geometry and particle physics.

The intended audience consists of mathematicians with some knowledge of particle physics, and of theoretical physicists with some mathematical background. Concerning the level of this textbook, for mathematicians I assume prerequisites on gauge theories at the level of, e.g., [1, 2], and recommend to first read the book [3] to really appreciate the last few chapters of this book on particle physics/the Standard Model. For physicists, I assume knowledge of some basic algebra, Hilbert space, and operator theory (e.g., [4, Chap. 2]), and Riemannian geometry (e.g., [5, 6]). This makes the book particularly suitable for a starting Ph.D. student, after a Master's degree in Mathematical/Theoretical Physics including the above background.

I would like to thank the organizers and participants of the aforementioned schools for their involvement and their feedback. This also applies to the MRI-Masterclass in Utrecht in 2010 and the Conference on index theory in Bogotá in 2008, where Chap. 5 finds its roots. Much feedback on previous drafts was gratefully received from students in my class on noncommutative geometry in Nijmegen: Bas Jordans, Joey van der Leer, and Sander Uijlen. I thank my students and co-authors Jord Boeijink, Thijs van den Broek, and Koen van den Dungen for allowing me to transcribe part of our results in the present book form. Simon Brain, Alan Carey, Roberta Iseppi, and Adam Rennie are gratefully acknowledged for their feedback and suggested corrections. Strong motivation to writing this book was given to me by my co-author Matilde Marcolli. I thank Gerard Bäuerle, Gianni Landi, and Klaas Landsman for having been my main tutors in writing, and Klaas in particular for a careful final proofreading. I also thank Aldo Rampioni at Springer for his help and guidance. I thank Alain Connes for his inspiration and enthusiasm for the field, without whose work this book could of course not have been written.

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References

1. Bleecker, D.: Gauge Theory and Variational Principles. Addison-wesley, Reading (1981)
2. Baez, J., Muniain, J.P.: Gauge fields, knots and gravity. Knots and Everything, vol. 4. World Scientific Publishing Co. Inc., River Edge (1994)
3. Cottingham, W.N., Greenwood, D.A.: An introduction to the standard model of particle physics, 2nd edn. Cambridge University Press, Cambridge (2007)
4. Thirring, W.: Quantum mathematical physics, 2nd edn. (Atoms, molecules and large systems, Translated from the 1979 and 1980 German originals by Evans M. Harrell II). Springer, Berlin (2002)
5. Jost, J.: Riemannian geometry and geometric analysis, 3rd edn. Universitext, Springer, Berlin (2002)
6. Nakahara, M.: Geometry, Topology and Physics. IOP Publishing, Bristol (1990)

Contents

Abbreviations and Symbols

$(\cdot,\cdot)$	Inner product (with values in $\mathbb{C}$)
$\langle\cdot,\cdot\rangle$	Hermitian structure/algebra valued inner product
$\|\cdot\|_{\mathrm{Lip}}$	Lipschitz (semi)-norm
$\|\cdot\|_s$	Sobolev norm
$\|\cdot\|$	C^*-norm
$\mathcal{A}$	*-algebra
A	C^*-algebra
$\mathcal{A}^\circ$ or A°	Opposite algebra
$\hat{A}$	Structure space of A
(A,H,D)	Finite spectral triple
$(A,H,D;J,\gamma)$	Finite real spectral triple
$(\mathcal{A},\mathcal{H},D)$	Spectral triple
$(\mathcal{A},\mathcal{H},D;J,\gamma)$	Real spectral triple
$\mathcal{A}_J$	Commutative subalgebra of $\mathcal{A}$
$\mathcal{A}^N$	Direct sum of N copies of $\mathcal{A}$
$\hat{A}(R)$	$\hat{A}$-form of Riemannian curvature
α	Algebra automorphism
$\mathrm{Aut}(\mathcal{A})$	Group of algebra automorphisms
A_μ	Gauge field
$\mathrm{ad}\,A_\mu$	Gauge field in adjoint representation
Ad	Adjoint action
a_k	Seeley–DeWitt coefficient
$\mathcal{B}(\mathcal{H})$	C^*-algebra of bounded operators on Hilbert space $\mathcal{H}$
$\mathfrak{B}$	C^*-bundle
b, B	Boundary operators on cochains
B_μ	Gauge field in adjoint representation
$C(X)$	C^*-algebra of continuous functions on compact topological space X
$\mathbb{C}^{n\circ}$	Defining representation of $M_n(\mathbb{C})^\circ$
$\mathrm{Cl}(V,Q)$	Clifford algebra
χ	$\mathbb{Z}_2$-grading on Clifford algebra
$\mathrm{Cl}^0(V,Q)$	Even part of Clifford algebra
$\mathrm{Cl}^1(V,Q)$	Odd part of Clifford algebra

$\mathrm{Cl}_n^{\pm}$	Clifford algebra $\mathrm{Cl}(\mathbb{R}^n, \pm Q_n)$
$\mathbb{C}\mathrm{l}_n$	Clifford algebra $\mathrm{Cl}(\mathbb{C}^n, Q_n)$
$\mathbb{C}\mathrm{l}_n^0$	Even part of Clifford algebra $\mathbb{C}\mathrm{l}_n$
$\mathbb{C}\mathrm{l}_n^1$	Odd part of Clifford algebra $\mathbb{C}\mathrm{l}_n$
$(\mathrm{Cl}_n^{\pm})^0$	Even part of Clifford algebra $\mathrm{Cl}_n^{\pm}$
$(\mathrm{Cl}_n^{\pm})^1$	Odd part of Clifford algebra $\mathrm{Cl}_n^{\pm}$
$\mathrm{Cl}^{\pm}(TM)$	Clifford algebra bundle
$\mathrm{Cliff}^{\pm}(M)$	Space of sections of Clifford algebra bundle
$\mathbb{C}\mathrm{l}(TM)$	Complexified Clifford algebra bundle
$\mathbb{C}\mathrm{liff}(M)$	Space of sections of complexified Clifford algebra bundle
c	Clifford multiplication
$C^n(\mathcal{A})$	n-cochains on algebra $\mathcal{A}$
$C^{\mathrm{ev}}(\mathcal{A})$	Even cocains
$C^{\mathrm{odd}}(\mathcal{A})$	Odd cochains
$c_{p,k}$	Combinatorial coefficients in residue cocycle
$c(k_1, \ldots, k_j)$	Combinatorial coefficients in residue cocycle
$\mathrm{ch}(E)$	Chern character of vector bundle E
$C_{\mu\nu\rho\sigma}$	Weyl curvature
$\chi(M)$	Euler characteristic
D	Self-adjoint operator
D_F	Finite Dirac operator
d_{ij}	Metric on finite discrete space
D_{ij}	Components of Dirac operator in finite spectral triple
$D_{ij,kl}$	Components of Dirac operator in finite real spectral triple
D_e	Component of finite Dirac operator labeling the edge e
D_M	Dirac operator on Riemannian spin manifold
Δ^E	Laplacian on vector bundle
Δ^S	Laplacian on spinor bundle
d_g	Riemannian distance function
$d(x, y)$	Distance function
d	Derivation given by $[D, \cdot]$
δ	Derivation given by $[\lvert D\rvert, \cdot]$
D_p	pDp
D_u	PuP with $P = \frac{1}{2}(1 + \mathrm{Sign}\, D)$
$\mathrm{Der}(\mathcal{A})$	Lie algebra of algebra derivations
D_ω	Operator D in the presence of inner fluctuation ω
d_R	Right-handed down quark
d_L	Left-handed down quark
$\mathcal{E}$ or E	Algebra module
$\mathcal{E}^{\circ}$ or E°	Conjugate module
E_A	Right A-module
${}_AE$	Left A-module
${}_AE_B$	$A - B$-bimodule

$E \otimes_A F$	Balanced tensor product
$\mathrm{End}_{\mathcal{A}}(\mathcal{E})$	Algebra of module endomorphisms
$\mathrm{End}(S)$	Endomorphism bundle of S
e_R	Right-handed electron
e_L	Left-handed electron
$\mathbb{F}$	Field ($\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$)
F_B	Right B-module
$F_B \circ_B E$	Kasparov product
ϕ_n	(b, B)-cocycle
$\langle \phi, p \rangle$	Pairing between even (b, B)-cocycle and projection
$\langle \phi, u \rangle$	Pairing between odd (b, B)-cocycle and unitary
$f[x_0, \ldots, x_n]$	Divided difference of f of order n
f_k	Moments of function f
F_X	Two-point space
F	Finite noncommutative space/finite real spectral triple
F_{ED}	Finite real spectral triple for electrodynamics
F_{YM}	Finite real spectral triple for Yang–Mills theory
F_{SM}	Finite real spectral triple for the Standard Model
$F_{\mu\nu}$	Field strength (curvature) of B_μ
ϕ, Φ	Scalar field
$\Gamma = (\Gamma^{(0)}, \Gamma^{(1)})$	Graph
(Γ, Λ)	Λ-decorated graph/Krajewski diagram
$\Gamma(E)$	Space of continuous sections of a vector bundle E
$\Gamma^\infty(E)$	Space of smooth sections of a vector bundle E
$\Gamma^\kappa_{\mu\nu}$	Christoffel symbols
$\widetilde{\Gamma}^b_{\mu a}$	Christoffel symbols in orthonormal basis
γ_{n+1}	Chirality operator
γ_μ	Dirac gamma matrices, generators of $\mathrm{Cliff}^+(M)$
γ_a	Generators of $\mathrm{Cliff}^+(M)$ in orthonormal basis
γ_M	$\mathbb{Z}_2$-grading on spinors
γ	$\mathbb{Z}_2$-grading
g	Riemannian metric
$\mathfrak{G}(\mathcal{A}, \mathcal{H}; J)$	Gauge group
$\mathfrak{G}(M \times F)$	Gauge group of almost-commutative manifold
$\mathfrak{G}(F)$	Gauge group of finite noncommutative space
$\mathfrak{g}(\mathcal{A}, \mathcal{H}; J)$	Gauge Lie algebra
$\mathfrak{g}(M \times F)$	Gauge Lie algebra of almost-commutative manifold
$\mathfrak{g}(F)$	Gauge Lie algebra of finite noncommutative space
G_μ	$SU(3)$ Standard Model gauge field
$G_{\mu\nu}$	Field strength (curvature) of G_μ
g_1	$U(1)$ coupling constant
g_2	$SU(2)$ coupling constant
g_3	$SU(3)$ coupling constant

$\mathbb{H}$	Quaternions
$\mathcal{H}(H)$	Hilbert space (finite-dimensional Hilbert space)
$\mathcal{H}^+$	Positive eigenspace of grading γ
$\mathcal{H}^s$	Sobolev space
$\mathcal{H}_{cl}$	Set of Grassmann variables in $\mathcal{H}$
H_l	Hilbert space of leptons
H_q	Hilbert space of quarks
$HH^n(\mathcal{A})$	Hochschild cohomology
$HCP^{\text{ev}}(\mathcal{A})$	Even cyclic cohomology
$HCP^{\text{odd}}(\mathcal{A})$	Odd cyclic cohomology
$\text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$	Space of module endomorphisms
$\mathfrak{H}(F)$	Group of unitary elements in commutative subalgebra
$\mathfrak{h}(F)$	Lie algebra of skew-hermitian elements in commutative subalgebra
H	Higgs field
h	Higgs field (unitary gauge)
$\mathbb{I}_N$	$N \times N$ identity matrix
$\text{Inn}(\mathcal{A})$	Group of inner automorphisms
J	Anti-unitary operator/real structure
$j : \Gamma \to \Gamma$	Involutive graph automorphism
$J_n^{\pm}$	Anti-linear map
J_M	Charge conjugation
$\text{KK}_f(A, B)$	Set of Kasparov modules for (A, B)
κ	Effective Yukawa coupling for tau-neutrino
$\mathcal{L}$	Lagrangian
Λ_μ	$U(1)$ Standard Model gauge field
$\Lambda_{\mu\nu}$	Field strength (curvature) of Λ_μ
λ	Higgs quartic coupling
Λ_{12}	Electroweak unification scale
$\mathit{\Lambda}_{23}$	Weak-strong unification scale
$\mathit{\Lambda}_{GUT}$	Grand unification scale
$M_n(\mathbb{F})$	*-algebra of $n \times n$ matrices with entries in $\mathbb{F}$
$M_n(A)$	*-algebra of $n \times n$ matrices with entries in A
M	Manifold
$M \times F$	Almost-commutative manifold
μ_N	Group of N'th roots of unity
M_W	Mass of W-boson
M_Z	Mass of Z-boson
m_{top}	Mass of top quark
m_h	Higgs mass
m_ν	Neutrino mass
m_R	Majorana mass matrix
m_l	Effective mass of tau-neutrino
$\boldsymbol{n}$	Defining representation $\mathbb{C}^n$ of $M_n(\mathbb{C})$

∇	Connection on module/vector bundle
$\nabla^{\mathfrak{B}}$	*-algebra connection
∇^S	Spin connection
ν_R	Right-handed neutrino
ν_L	Left-handed neutrino
ω	Connection one-form/inner fluctuation
$\omega^\#$	Vector field corresponding to one-form ω
Ω^E	Curvature of connection on bundle E
$\Omega^k_{\mathrm{dR}}(M)$	De Rham differential k-forms
$\Omega^k_{\mathrm{dR}}(M, A)$	De Rham differential k-forms with values in algebra A
$\Omega_k(M)$	De Rham$k$-currents on M
$\Omega^1_D(\mathcal{A})$	Connes' differential one-forms associated to a spectral triple
op^r	Space of operators of analytic order $\leq r$
$\mathrm{Out}(\mathcal{A})$	Group of outer automorphisms
π	Algebra representation
$\pi(A)'$	Commutant of $\pi(A)$
$\Psi_p(a^0, \ldots, a^p)$	Improper (b, B)-cocycle
$\Psi(\mathcal{A})$	Pseudodifferential operators
$\Psi^k(\mathcal{A})$	Pseudodifferential operators of order k
p	Projection
$PU(N)$	Projective unitary group
q	Quaternion
q_λ	Embedding of $\mathbb{C}$ in $\mathbb{H}$
Q	Quadratic form
Q_n	Standard quadratic form on $\mathbb{R}^n$ or $\mathbb{C}^n$
Q_μ	$SU(2)$ Standard Model gauge field
$Q_{\mu\nu}$	Field strength (curvature) of Q_μ
$R(X, Y)$	Riemannian curvature tensor
$R_{\mu\nu\kappa\lambda}$	Riemannian curvature tensor
$R_{\nu\lambda}$	Ricci tensor
R^*R^*	Pontryagin class
$\mathrm{res}_{s=0}\Psi_p$	Residue cocycle
ρ	Ratio between mass of tau-neutrino and top quark
s	Scalar curvature
S	Spinor bundle
Sd	Dimension spectrum
$S_b[\omega]$	Spectral action
$S_b^{(n)}$	n'th Gâteaux derivative of spectral action
$S_{top}[\omega]$	Topological spectral action
$S_f[\omega, \psi]$	Fermionic action
$S\mathcal{U}(\mathcal{A}_F)$	Group of elements in $\mathcal{U}(\mathcal{A}_F)$ with determinant 1
$S\mathfrak{H}(F)$	Group of elements in $\mathfrak{H}(F)$ with determinant 1
$\mathfrak{su}(\mathcal{A}_F)$	Lie algebra of traceless elements in $\mathfrak{u}(\mathcal{A}_F)$

$s\mathfrak{h}(F)$	Lie algebra of traceless elements in $\mathfrak{h}(F)$
S_N	Group of permutations on N elements
$SO(N)$	Group of special orthogonal transformations of $\mathbb{R}^N$
$SU(N)$	Group of special unitary transformations of $\mathbb{C}^N$
$su(N)$	Lie algebra of traceless skew-hermitian transformations of $\mathbb{C}^N$
σ	Real scalar field
TM	Tangent bundle
T^*M	Cotangent bundle
$T^{(k)}$	k'th iteration of derivation $T \mapsto [D^2, T]$
u	Unitary algebra element
U	Unitary operator
$\mathcal{U}(\mathcal{A})$	Group of unitary elements in *-algebra $\mathcal{A}$
$\mathfrak{u}(\mathcal{A})$	Lie algebra of skew-hermitian algebra elements
$U(N)$	Group of unitary transformations of $\mathbb{C}^N$
$u(N)$	Lie algebra of skew-hermitian transformations of $\mathbb{C}^N$
u_R	Right-handed up quark
u_L	Left-handed up quark
V_μ	$SU(3)$ Standard Model gauge field
$V_{\mu\nu}$	Field strength (curvature) of V_μ
v	Higgs vacuum expectation value
W_μ	$SU(2)$ Standard Model gauge field
$W_{\mu\nu}$	Field strength (curvature) of W_μ
$\langle X^0, \ldots, X^p \rangle_z$	Zeta functions defining the improper cocycles
Y_ν, Y_e, Y_u, Y_d	Yukawa mass matrices
Y_R	Majorana mass matrix
Y_μ	$U(1)$ Standard Model gauge field
$Y_{\mu\nu}$	Field strength (curvature) of Y_μ
y_ν	Yukawa coupling of tau-neutrino
y_{top}	Yukawa coupling of top quark
$Z(\mathcal{A})$	Center of $\mathcal{A}$
$\zeta(z)$	Riemann zeta function
$\zeta_E(z)$	Epstein zeta function
$\zeta_b(z)$	Zeta function