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Walter D. van Suijlekom

# Noncommutative Geometry and Particle Physics

 Springer

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The Netherlands

ISSN 0921-3767                      ISSN 2352-3905 (electronic)  
ISBN 978-94-017-9161-8            ISBN 978-94-017-9162-5 (eBook)  
DOI 10.1007/978-94-017-9162-5  
Springer Dordrecht Heidelberg New York London

Library of Congress Control Number: 2014939937

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# Preface

The seeds of this book have been planted in the far east, where I wrote lecture notes for international schools in Tianjin, China in 2007, and in Bangkok, Thailand in 2011. I then realized that an up-to-date text for beginning noncommutative geometers on the applications of this rather new mathematical field to particle physics was missing in the literature.

This made me decide to transform my notes into the form of a book. Besides the given challenge inherent in such a project, this was not made easy because of recent, rapid developments in the field, making it difficult to choose what to include and to decide where to stop in my treatment. The current state of affairs is at least touched upon in the final chapter of this book, where I discuss the latest particle physics models in noncommutative geometry, and compare them to the latest experimental findings. With this, I hope to have provided a path that starts with the basic principles of noncommutative geometry and leads to the forefront of research in noncommutative geometry and particle physics.

The intended audience consists of mathematicians with some knowledge of particle physics, and of theoretical physicists with some mathematical background. Concerning the level of this textbook, for mathematicians I assume prerequisites on gauge theories at the level of, e.g., [1, 2], and recommend to first read the book [3] to really appreciate the last few chapters of this book on particle physics/the Standard Model. For physicists, I assume knowledge of some basic algebra, Hilbert space, and operator theory (e.g., [4, Chap. 2]), and Riemannian geometry (e.g., [5, 6]). This makes the book particularly suitable for a starting Ph.D. student, after a Master's degree in Mathematical/Theoretical Physics including the above background.

I would like to thank the organizers and participants of the aforementioned schools for their involvement and their feedback. This also applies to the MRI-Masterclass in Utrecht in 2010 and the Conference on index theory in Bogotá in 2008, where [Chap. 5](#) finds its roots. Much feedback on previous drafts was gratefully received from students in my class on noncommutative geometry in Nijmegen: Bas Jordans, Joey van der Leer, and Sander Uijlen. I thank my students and co-authors Jord Boeijink, Thijs van den Broek, and Koen van den Dungen for allowing me to transcribe part of our results in the present book form. Simon Brain, Alan Carey, Roberta Iseppi, and Adam Rennie are gratefully acknowledged for their feedback and suggested corrections. Strong motivation to writing this

book was given to me by my co-author Matilde Marcolli. I thank Gerard Bäuerle, Gianni Landi, and Klaas Landsman for having been my main tutors in writing, and Klaas in particular for a careful final proofreading. I also thank Aldo Rampioni at Springer for his help and guidance. I thank Alain Connes for his inspiration and enthusiasm for the field, without whose work this book could of course not have been written.

I am thankful to my family and friends for their continuous love and support. My deepest gratitude goes to Mathilde for being my companion in life, and to Daniël for making sure that the final stages of writing were frequently, and happily, interrupted.

April 2014

Walter D. van Suijlekom

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# Abbreviations and Symbols

$(\cdot, \cdot)$	Inner product (with values in $\mathbb{C}$ )
$\langle \cdot, \cdot \rangle$	Hermitian structure/algebra valued inner product
$\  \cdot \ _{\text{Lip}}$	Lipschitz (semi)-norm
$\  \cdot \ _s$	Sobolev norm
$\  \cdot \ $	$C^*$ -norm
$\mathcal{A}$	*-algebra
$A$	$C^*$ -algebra
$\mathcal{A}^\circ$ or $\hat{A}$	Opposite algebra
$\hat{A}$	Structure space of $A$
$(A, H, D)$	Finite spectral triple
$(A, H, D; J, \gamma)$	Finite real spectral triple
$(\mathcal{A}, \mathcal{H}, D)$	Spectral triple
$(\mathcal{A}, \mathcal{H}, D; J, \gamma)$	Real spectral triple
$\mathcal{A}_J$	Commutative subalgebra of $\mathcal{A}$
$\mathcal{A}^N$	Direct sum of $N$ copies of $\mathcal{A}$
$\hat{A}(R)$	$\hat{A}$ -form of Riemannian curvature
$\alpha$	Algebra automorphism
$\text{Aut}(\mathcal{A})$	Group of algebra automorphisms
$A_\mu$	Gauge field
$\text{ad} A_\mu$	Gauge field in adjoint representation
$\text{Ad}$	Adjoint action
$a_k$	Seeley–DeWitt coefficient
$\mathcal{B}(\mathcal{H})$	$C^*$ -algebra of bounded operators on Hilbert space $\mathcal{H}$
$\mathfrak{B}$	$C^*$ -bundle
$b, B$	Boundary operators on cochains
$B_\mu$	Gauge field in adjoint representation
$C(X)$	$C^*$ -algebra of continuous functions on compact topological space $X$
$\mathbb{C}^{n^\circ}$	Defining representation of $M_n(\mathbb{C})^\circ$
$\text{Cl}(V, Q)$	Clifford algebra
$\chi$	$\mathbb{Z}_2$ -grading on Clifford algebra
$\text{Cl}^0(V, Q)$	Even part of Clifford algebra
$\text{Cl}^1(V, Q)$	Odd part of Clifford algebra

$\text{Cl}_n^\pm$	Clifford algebra $\text{Cl}(\mathbb{R}^n, \pm Q_n)$
$\text{Cl}_n$	Clifford algebra $\text{Cl}(\mathbb{C}^n, Q_n)$
$\text{Cl}_n^0$	Even part of Clifford algebra $\text{Cl}_n$
$\text{Cl}_n^1$	Odd part of Clifford algebra $\text{Cl}_n$
$(\text{Cl}_n^\pm)^0$	Even part of Clifford algebra $\text{Cl}_n^\pm$
$(\text{Cl}_n^\pm)^1$	Odd part of Clifford algebra $\text{Cl}_n^\pm$
$\text{Cl}^\pm(TM)$	Clifford algebra bundle
$\text{Cliff}^\pm(M)$	Space of sections of Clifford algebra bundle
$\text{Cl}(TM)$	Complexified Clifford algebra bundle
$\text{Cliff}(M)$	Space of sections of complexified Clifford algebra bundle
$c$	Clifford multiplication
$C^n(\mathcal{A})$	$n$ -cochains on algebra $\mathcal{A}$
$C^{\text{ev}}(\mathcal{A})$	Even cocains
$C^{\text{odd}}(\mathcal{A})$	Odd cocains
$c_{p,k}$	Combinatorial coefficients in residue cocycle
$c(k_1, \dots, k_j)$	Combinatorial coefficients in residue cocycle
$\text{ch}(E)$	Chern character of vector bundle $E$
$C_{\mu\nu\rho\sigma}$	Weyl curvature
$\chi(M)$	Euler characteristic
$D$	Self-adjoint operator
$D_F$	Finite Dirac operator
$d_{ij}$	Metric on finite discrete space
$D_{ij}$	Components of Dirac operator in finite spectral triple
$D_{ij,kl}$	Components of Dirac operator in finite real spectral triple
$D_e$	Component of finite Dirac operator labeling the edge $e$
$D_M$	Dirac operator on Riemannian spin manifold
$\Delta^E$	Laplacian on vector bundle
$\Delta^S$	Laplacian on spinor bundle
$d_g$	Riemannian distance function
$d(x, y)$	Distance function
$d$	Derivation given by $[D, \cdot]$
$\delta$	Derivation given by $[ D , \cdot]$
$D_p$	$pDp$
$D_u$	$PuP$ with $P = \frac{1}{2}(1 + \text{Sign } D)$
$\text{Der}(\mathcal{A})$	Lie algebra of algebra derivations
$D_\omega$	Operator $D$ in the presence of inner fluctuation $\omega$
$d_R$	Right-handed down quark
$d_L$	Left-handed down quark
$\mathcal{E}$ or $E$	Algebra module
$\mathcal{E}^\circ$ or $E^\circ$	Conjugate module
$E_A$	Right $A$ -module
${}_A E$	Left $A$ -module
${}_A E_B$	$A - B$ -bimodule

$E \otimes_A F$	Balanced tensor product
$\text{End}_{\mathcal{A}}(\mathcal{E})$	Algebra of module endomorphisms
$\text{End}(S)$	Endomorphism bundle of $S$
$e_R$	Right-handed electron
$e_L$	Left-handed electron
$\mathbb{F}$	Field ( $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ )
$F_B$	Right $B$ -module
$F_B \circ_B E$	Kasparov product
$\phi_n$	$(b, B)$ -cocycle
$\langle \phi, p \rangle$	Pairing between even $(b, B)$ -cocycle and projection
$\langle \phi, u \rangle$	Pairing between odd $(b, B)$ -cocycle and unitary
$f[x_0, \dots, x_n]$	Divided difference of $f$ of order $n$
$f_k$	Moments of function $f$
$F_X$	Two-point space
$F$	Finite noncommutative space/finite real spectral triple
$F_{ED}$	Finite real spectral triple for electrodynamics
$F_{YM}$	Finite real spectral triple for Yang–Mills theory
$F_{SM}$	Finite real spectral triple for the Standard Model
$F_{\mu\nu}$	Field strength (curvature) of $B_\mu$
$\phi, \Phi$	Scalar field
$\Gamma = (\Gamma^{(0)}, \Gamma^{(1)})$	Graph
$(\Gamma, \Lambda)$	$\Lambda$ -decorated graph/Krajewski diagram
$\Gamma(E)$	Space of continuous sections of a vector bundle $E$
$\Gamma^\infty(E)$	Space of smooth sections of a vector bundle $E$
$\Gamma_{\mu\nu}^\kappa$	Christoffel symbols
$\tilde{\Gamma}_{\mu a}^b$	Christoffel symbols in orthonormal basis
$\gamma_{n+1}$	Chirality operator
$\gamma_\mu$	Dirac gamma matrices, generators of $\text{Cliff}^+(M)$
$\gamma_a$	Generators of $\text{Cliff}^+(M)$ in orthonormal basis
$\gamma_M$	$\mathbb{Z}_2$ -grading on spinors
$\gamma$	$\mathbb{Z}_2$ -grading
$g$	Riemannian metric
$\mathfrak{G}(\mathcal{A}, \mathcal{H}; J)$	Gauge group
$\mathfrak{G}(M \times F)$	Gauge group of almost-commutative manifold
$\mathfrak{G}(F)$	Gauge group of finite noncommutative space
$\mathfrak{g}(\mathcal{A}, \mathcal{H}; J)$	Gauge Lie algebra
$\mathfrak{g}(M \times F)$	Gauge Lie algebra of almost-commutative manifold
$\mathfrak{g}(F)$	Gauge Lie algebra of finite noncommutative space
$G_\mu$	$SU(3)$ Standard Model gauge field
$G_{\mu\nu}$	Field strength (curvature) of $G_\mu$
$g_1$	$U(1)$ coupling constant
$g_2$	$SU(2)$ coupling constant
$g_3$	$SU(3)$ coupling constant

$\mathbb{H}$	Quaternions
$\mathcal{H}(H)$	Hilbert space (finite-dimensional Hilbert space)
$\mathcal{H}^+$	Positive eigenspace of grading $\gamma$
$\mathcal{H}^s$	Sobolev space
$\mathcal{H}_{cl}$	Set of Grassmann variables in $\mathcal{H}$
$H_l$	Hilbert space of leptons
$H_q$	Hilbert space of quarks
$HH^n(\mathcal{A})$	Hochschild cohomology
$HCP^{ev}(\mathcal{A})$	Even cyclic cohomology
$HCP^{odd}(\mathcal{A})$	Odd cyclic cohomology
$\text{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{F})$	Space of module endomorphisms
$\mathfrak{S}(F)$	Group of unitary elements in commutative subalgebra
$\mathfrak{h}(F)$	Lie algebra of skew-hermitian elements in commutative subalgebra
$H$	Higgs field
$h$	Higgs field (unitary gauge)
$\mathbb{I}_N$	$N \times N$ identity matrix
$\text{Inn}(\mathcal{A})$	Group of inner automorphisms
$J$	Anti-unitary operator/real structure
$j : \Gamma \rightarrow \Gamma$	Involutive graph automorphism
$J_n^\pm$	Anti-linear map
$J_M$	Charge conjugation
$\text{KK}_f(A, B)$	Set of Kasparov modules for $(A, B)$
$\kappa$	Effective Yukawa coupling for tau-neutrino
$\mathcal{L}$	Lagrangian
$\Lambda_\mu$	$U(1)$ Standard Model gauge field
$\Lambda_{\mu\nu}$	Field strength (curvature) of $\Lambda_\mu$
$\lambda$	Higgs quartic coupling
$\Lambda_{12}$	Electroweak unification scale
$\Lambda_{23}$	Weak-strong unification scale
$\Lambda_{GUT}$	Grand unification scale
$M_n(\mathbb{F})$	*-algebra of $n \times n$ matrices with entries in $\mathbb{F}$
$M_n(A)$	*-algebra of $n \times n$ matrices with entries in $A$
$M$	Manifold
$M \times F$	Almost-commutative manifold
$\mu_N$	Group of $N$ 'th roots of unity
$M_W$	Mass of $W$ -boson
$M_Z$	Mass of $Z$ -boson
$m_{top}$	Mass of top quark
$m_h$	Higgs mass
$m_\nu$	Neutrino mass
$m_R$	Majorana mass matrix
$m_l$	Effective mass of tau-neutrino
$\mathbf{n}$	Defining representation $\mathbb{C}^n$ of $M_n(\mathbb{C})$

$\nabla$	Connection on module/vector bundle
$\nabla^{\mathfrak{B}}$	*-algebra connection
$\nabla^S$	Spin connection
$\nu_R$	Right-handed neutrino
$\nu_L$	Left-handed neutrino
$\omega$	Connection one-form/inner fluctuation
$\omega^\#$	Vector field corresponding to one-form $\omega$
$\Omega^E$	Curvature of connection on bundle $E$
$\Omega_{\text{dR}}^k(M)$	De Rham differential $k$ -forms
$\Omega_{\text{dR}}^k(M, A)$	De Rham differential $k$ -forms with values in algebra $A$
$\Omega_k(M)$	De Rham $k$ -currents on $M$
$\Omega_D^1(\mathcal{A})$	Connes' differential one-forms associated to a spectral triple
$\text{op}^r$	Space of operators of analytic order $\leq r$
$\text{Out}(\mathcal{A})$	Group of outer automorphisms
$\pi$	Algebra representation
$\pi(A)'$	Commutant of $\pi(A)$
$\Psi_p(a^0, \dots, a^p)$	Improper $(b, B)$ -cocycle
$\Psi(\mathcal{A})$	Pseudodifferential operators
$\Psi^k(\mathcal{A})$	Pseudodifferential operators of order $k$
$p$	Projection
$PU(N)$	Projective unitary group
$q$	Quaternion
$q_\lambda$	Embedding of $\mathbb{C}$ in $\mathbb{H}$
$Q$	Quadratic form
$Q_n$	Standard quadratic form on $\mathbb{R}^n$ or $\mathbb{C}^n$
$Q_\mu$	$SU(2)$ Standard Model gauge field
$Q_{\mu\nu}$	Field strength (curvature) of $Q_\mu$
$R(X, Y)$	Riemannian curvature tensor
$R_{\mu\nu\kappa\lambda}$	Riemannian curvature tensor
$R_{\nu\lambda}$	Ricci tensor
$R^*R^*$	Pontryagin class
$\text{res}_{s=0}\Psi_p$	Residue cocycle
$\rho$	Ratio between mass of tau-neutrino and top quark
$s$	Scalar curvature
$S$	Spinor bundle
$\text{Sd}$	Dimension spectrum
$S_b[\omega]$	Spectral action
$S_b^{(n)}$	$n$ 'th Gâteaux derivative of spectral action
$S_{\text{top}}[\omega]$	Topological spectral action
$S_f[\omega, \psi]$	Fermionic action
$SU(\mathcal{A}_F)$	Group of elements in $\mathcal{U}(\mathcal{A}_F)$ with determinant 1
$S\mathfrak{H}(F)$	Group of elements in $\mathfrak{H}(F)$ with determinant 1
$\text{su}(\mathcal{A}_F)$	Lie algebra of traceless elements in $\mathfrak{u}(\mathcal{A}_F)$

$\mathfrak{sb}(F)$	Lie algebra of traceless elements in $\mathfrak{h}(F)$
$S_N$	Group of permutations on $N$ elements
$SO(N)$	Group of special orthogonal transformations of $\mathbb{R}^N$
$SU(N)$	Group of special unitary transformations of $\mathbb{C}^N$
$\mathfrak{su}(N)$	Lie algebra of traceless skew-hermitian transformations of $\mathbb{C}^N$
$\sigma$	Real scalar field
$TM$	Tangent bundle
$T^*M$	Cotangent bundle
$T^{(k)}$	$k$ 'th iteration of derivation $T \mapsto [D^2, T]$
$u$	Unitary algebra element
$U$	Unitary operator
$U(\mathcal{A})$	Group of unitary elements in $*$ -algebra $\mathcal{A}$
$\mathfrak{u}(\mathcal{A})$	Lie algebra of skew-hermitian algebra elements
$U(N)$	Group of unitary transformations of $\mathbb{C}^N$
$\mathfrak{u}(N)$	Lie algebra of skew-hermitian transformations of $\mathbb{C}^N$
$u_R$	Right-handed up quark
$u_L$	Left-handed up quark
$V_\mu$	$SU(3)$ Standard Model gauge field
$V_{\mu\nu}$	Field strength (curvature) of $V_\mu$
$v$	Higgs vacuum expectation value
$W_\mu$	$SU(2)$ Standard Model gauge field
$W_{\mu\nu}$	Field strength (curvature) of $W_\mu$
$\langle X^0, \dots, X^p \rangle_z$	Zeta functions defining the improper cocycles
$Y_v, Y_e, Y_u, Y_d$	Yukawa mass matrices
$Y_R$	Majorana mass matrix
$Y_\mu$	$U(1)$ Standard Model gauge field
$Y_{\mu\nu}$	Field strength (curvature) of $Y_\mu$
$y_\nu$	Yukawa coupling of tau-neutrino
$y_{top}$	Yukawa coupling of top quark
$Z(\mathcal{A})$	Center of $\mathcal{A}$
$\zeta(z)$	Riemann zeta function
$\zeta_E(z)$	Epstein zeta function
$\zeta_b(z)$	Zeta function