White Noise

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# White Noise

## An Infinite Dimensional Calculus

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## Preface

Many areas of applied mathematics call for an efficient calculus in infinite dimensions. This is most apparent in quantum physics and in all disciplines of science which describe natural phenomena by equations involving stochasticity. With this monograph we intend to provide a framework for analysis in infinite dimensions which is flexible enough to be applicable in many areas, and which on the other hand is intuitive and efficient. Whether or not we achieved our aim must be left to the judgment of the reader.

This book treats the theory and applications of analysis and functional analysis in infinite dimensions based on *white noise*. By white noise we mean the generalized Gaussian process which is (informally) given by the time derivative of the Wiener process, i.e., by the velocity of Brownian motion. Therefore, in essence we present analysis on a Gaussian space, and applications to various areas of science.

Calculus, analysis, and functional analysis in infinite dimensions (or dimension-free formulations of these parts of classical mathematics) have a long history. Early examples can be found in the works of Dirichlet, Euler, Hamilton, Lagrange, and Riemann on variational problems. At the beginning of this century, Fréchet, Gâteaux and Volterra made essential contributions to the calculus of functions over infinite dimensional spaces. The important and inspiring work of Wiener and Lévy followed during the first half of this century. Moreover, the articles and books of Wiener and Lévy had a view towards probability theory.

Almost from its very beginning quantum physics has been using infinite dimensional calculus and it became an important (though sometimes informal) tool in modern theoretical and mathematical physics through the pioneering work of Berezin, Cook, Feynman, Fock, Schwinger, and Segal. During the sixties and seventies, in the context of Euclidian and constructive quantum field theory, many structures and deep results in infinite dimensional analysis have been worked out by Albeverio and Høegh-Krohn, Glimm and Jaffe, Gross, Nelson, Simon, and

Symanzik (to name just a few). In parallel, first approaches to a systematic (functional) analysis over Gaussian spaces have been developed. Among others, there was the influential work of Berezanskii and Kondratiev, Gross, Krée, and Malliavin, and their respective schools. The approach which is presented in this book has its roots in a paper by Hida in 1975, and its essential difference to other formulations is the specific choice of the white noise space as the underlying Gaussian space.

This choice is motivated by the following facts. On one hand white noise is a very simple process: it is stationary centered Gaussian, and it is independent at different moments of time. The combination of these properties singles it out as a natural object to start from: in a sense, white noise is a "universal" source of randomness, and as such it appears in many fields of applications. In particular, the independence at different moments of time suggests an interpretation of white noise as a (by the time parameter) continuously indexed coordinate system in infinite dimensions.

The theoretical core of this book consists of the first five chapters. In the first two chapters we present some general facts concerning Gaussian spaces and integrable random variables. We shall review there the Wiener–Itô chaos decomposition theorem which will play an essential role as we develop the analysis of functions of white noise. In the third chapter we introduce various (classes of) spaces of smooth and generalized functionals over a Gaussian space. The fourth chapter is devoted to the detailed study of one specific dual pair of spaces of smooth and generalized random variables over the white noise space. Chapter five deals with the theory of differential operators for functions of white noise.

In the remainder of this book, the framework which is formulated in the first five chapters is developed in the directions of harmonic and stochastic analysis. Furthermore, there are chapters with applications in potential theory and mathematical physics. In a number of appendices we collected some (more or less well–known) facts which the reader might find useful.

Large parts of the first three chapters are written for general Gaussian spaces which are given by the dual of a nuclear space equipped with a Gaussian measure, because this generality comes here at no extra costs. However, from the fourth chapter on, we work almost exclusively with the choice of the white noise probability space. On one hand, many of the structures which are developed need more than just a general Gaussian space as the underlying probability space. (For example, it is sometimes necessary that the sample space consists of (generalized) functions.) On the other hand: although often structures can be formulated and results hold for more general set-ups, we felt that it is more transparent and economical to present them in the white noise case.

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