

## DISPLAYING MODAL LOGIC

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## PREFACE

The present monograph is a slightly revised version of my Habilitationsschrift *Proof-theoretic Aspects of Intensional and Non-Classical Logics*, successfully defended at Leipzig University, November 1997. It collects work on proof systems for modal and constructive logics I have done over the last few years. The main concern is display logic, a certain refinement of Gentzen's sequent calculus developed by Nuel D. Belnap.

This book is far from offering a *comprehensive presentation* of generalized sequent systems for modal logics broadly conceived. The proof-theory of non-classical logics is a rapidly developing field, and even the generalizations of the ordinary notion of sequent listed in Chapter 1 can hardly be presented in great detail within a single volume. In addition to further investigating the various approaches toward generalized Gentzen systems, it is important to compare them and to discuss their relative advantages and disadvantages. An initial attempt at bringing together work on different kinds of proof systems for modal logics has been made in [188]. Another step in the same direction is [196]. Since Chapter 1 contains introductory considerations and, moreover, every remaining chapter begins with some surveying or summarizing remarks, in this preface I shall only emphasize a relation to philosophy that is important to me, register the sources of papers that have entered this book in some form or another, and acknowledge advice and support.

It may be difficult to immediately perceive a relation to a major *philosophical* problem in the present work, which is formal, although the proof-theoretic detail sometimes overemphasizes the technical aspects. Nevertheless, this book is not only meant as a contribution to logic, but also to philosophy. Due to the diversity of the philosophical subdisciplines, every general characterization of philosophy is bound to be quite abstract. According to some philosophers, the very aim of philosophy is to distinguish meaningful discourse from meaningless discourse about all kinds of problems and phenomena. More concretely, the aim of logic is often said to be separating the good arguments from the bad ones. From this perspective, logic plays a fundamental role both in philosophy and in scientific inquiry in general, for it is the meaning of the logical operations that is central to determining

the sound arguments and the meaningfulness of large parts of discourse. Logical semantics has many faces. In addition to the prevailing paradigm, namely ‘realistic’, model-theoretic semantics, there are also, for instance, algebraic semantics, game-theoretic semantics, dynamic semantics, and proof-theoretic semantics. In proof-theoretic semantics the meaning of the logical operations is specified in terms of general introduction schemata. They provide a defining framework in the sense that every permissible introduction rule is (or is interchangeable with) an instantiation of such a schema. Different generalizations of the standard notion of sequent suggest different proof-theoretic semantics, and in this book I give, among other things, a proof-theoretic characterization of the tense logical operations and prove other functional completeness theorems. I hope the philosophically inclined reader will regard these discussions and results as they are intended, namely as a contribution to the philosophy of meaning. In particular, it is shown that the idea of meaning-as-use is formally feasible also for modal and tense logics. A general discussion of the proof-theoretic approach to meaning, however, is beyond the scope of this book.

Various papers dissolved into parts of this work or became chapters of it in a slightly or more substantially revised form. In particular, the notation and terminology of these papers has also been harmonized, the internal coherence and interdependencies have been highlighted, and new material has been included. To be precise, this earlier work has been distributed as follows:

[181]	Chapters 1, 2, and 3	[185]	Chapter 4
[187]	Chapter 5	[191]	Chapter 6
[182]	Chapter 7	[183]	Chapter 8
[184], [192]	Chapter 9	[190]	Chapter 10
[193]	Chapter 11	[194]	Chapter 12
[189]	Section 13.1	[186]	Section 13.2

I would like to acknowledge inspiration from and discussions with various colleagues and friends. First of all, I wish to thank Nuel Belnap for having developed display logic and for his advice and encouragement. In particular, I am grateful to him for inviting me to Pittsburgh in 1994, for giving me the opportunity to present the material of Chapter 5 at the Philosophy Department of Pittsburgh University, for discussing strong cut-elimination with me in detail, and for various conversations on first-order display logic when he was a visiting Leibniz Professor at the Center for Advanced Studies of Leipzig University in 1996. Another source of inspiration has been Dov Gabbay. The work on struc-

tured consequence relations, semantics-based nonmonotonic reasoning, temporalization, and negation presented here takes as its starting point concepts developed by him. Moreover, my thinking about logic has been greatly influenced by Johan van Benthem's work. The lecture on first-order logic as modal logic he gave at Leipzig University in 1994 triggered the formulation of first-order display logic presented in Chapter 12. I would like to thank Johan van Benthem and Grigori Mints for inviting me to Stanford in 1994, Dov Gabbay for inviting me to London in 1996, Valentin Shehtman for inviting me to Moscow in 1996, and Makoto Kanazawa for inviting me to Chiba, also in 1996. These visits helped me a lot in thinking about display logic. As to the investigation of display logic, I am also greatly indebted to Rajeev Goré, Marcus Kracht and Greg Restall. I have learnt a lot from their beautiful work, and in particular Marcus Kracht commented upon preliminary drafts of various chapters. I wish to thank Siegfried Gottwald for providing a stimulating working environment at the Institute of Logic and Philosophy of Science at Leipzig University and for his supportive attitude toward my proof-theoretic interests. Moreover, I gratefully acknowledge useful comments on various parts of the present book by Seiki Akama, Tijn Borghuis, Kosta Došen, Jan Jaspars, Grigori Mints, David Pearce, Maarten de Rijke, Valentin Shehtman, and Yde Venema. Of course, I take full responsibility for all remaining defects and idiosyncrasy.

Finally, I wish to thank Petra, Kasimir, Friederike, Agnes Dakota, and Carlotta for the most enjoyable family life I can imagine. I gratefully dedicate this book to the memory of my mother, Agnes Wansing.

*Leipzig, March 1998*  
*Heinrich Wansing*