INTEGRABLE PROBLEMS OF CELESTIAL MECHANICS IN SPACES OF CONSTANT CURVATURE

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INTEGRABLE PROBLEMS OF CELESTIAL MECHANICS IN SPACES OF CONSTANT CURVATURE

by

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Introduction

The problem of integrability or nonintegrability of dynamical systems is one of the central problems of mathematics and mechanics. Integrable cases are of considerable interest, since, by examining them, one can study general laws of behavior for the solutions of these systems. The classical approach to studying dynamical systems assumes a search for explicit formulas for the solutions of motion equations and then their analysis. This approach stimulated the development of new areas in mathematics, such as the algebraic integration and the theory of elliptic and theta functions. In spite of this, the qualitative methods of studying dynamical systems are much actual.

It was Poincare who founded the qualitative theory of differential equations. Poincare, working out qualitative methods, studied the problems of celestial mechanics and cosmology in which it is especially important to understand the behavior of trajectories of motion, i.e., the solutions of differential equations at infinite time. Namely, beginning from Poincare systems of equations (in connection with the study of the problems of celestial mechanics), the right-hand parts of which don't depend explicitly on the independent variable of time, i.e., dynamical systems, are studied.

At present time the qualitative methods of studying dynamical systems, in particular, Hamiltonian systems, are actively being developed. According to the point of view fully formed in recent times, one of the basic qualitative characteristics of an integrable Hamiltonian system is a structure of the Liouville foliation, i.e., the foliation of phase space into the union of n-dimensional invariant Liouville tori (or cylinders), and, possibly, some singular fibers (some singular integral submanifolds). A.T. Fomenko suggested a new approach to studying this structure that allowed to construct the full topological invariant (Fomenko-Ziecshang invariant). On the basis of the suggested approach in the qualitative theory of integrable Hamiltonian systems the classification of the Liouville foliations on three-dimensional isoenergy surfaces is constructed. In this work the technique of the topological analyses is also applied.

For a long time, the three-body problem was considered as an object of investigation for many mathematicians and physicists. This famous problem can be formulated as follows. Three material points are mutually attracted according to the Newton law which reads that every two of these points attract each other with a force that is directly proportional to the masses of these points and inversely proportional to the squared distance separating them; the points can be situated in an arbitrary initial position and are free to move in space.

Up to now, the three-body problem has not been solved in the general form.

Euler was the first scientist to consider the problem of motion of a material point in the Euclidean space under the action of the field generated by two fixed Newtonian centers (a special case of the three-body problem). In general, virtually all the achievements of present-day celestial mechanics are based on the ideas of Euler, who worked fruitfully not only in this field of study but in others as well. As S.V.Vavilov put it, the mathematical brilliance of Euler was not matched by his physical intuition; that is why Euler the physicist was suppressed by Euler the mathematician. Nevertheless, the works by Euler in astronomy demonstrate his profound intuition and professionalism.

It was Euler who reduced the problem of two fixed centers to the quadratures. Unfortunately, the results obtained by Euler were of only theoretical importance in those days, because two fixed gravitating centers is a system that cannot be implemented in nature.

Euler did not examine integrals of motion in detail. He was stopped by the fact that he did not see any prospects for the application of this problem. The problems of the Sun-Jupiter-Saturn type cannot serve as an adequate model, since they do not satisfy the conditions of motion in the field of two fixed centers. And it was namely the applied aspect of this problem that was of primary importance for Euler the astronomer. (For Euler as a mathematician, it was less important, because pure mathematicians are often not interested in the applied meaning of a problem being solved, they are interested in abstract problems as such and want to obtain the result irrespective of possible applications.)

In the paper of E.P. Aksenov, E.A. Grebennikov, and V.G. Demin [23], a rather unexpected field of application of the Euler problem was found; namely, this problem was extended to complex values of parameters, that is, to the motion of artificial satellites in the gravity field of nonspherical planets. But Earth is just a nonspherical planet; it can be considered as two imaginary centers rather than a single center. In addition, one can examine the motion of a spacecraft in the gravity fields of two planets, neglecting the displacement of these planets over the time interval of the spacecraft's flight. Thus, the sphere of application was found, and the problem started its second life.

Another aspect of this problem is the motion of a mass point under the action of Newtonian attraction of a fixed center and a force which is constant in magnitude and direction (homogeneous field). This problem is the limiting case of the two-center problem, when one of the centers is moved to infinity in the force direction (its mass tends to infinity in such a way that the perturbing acceleration is constant, i.e., the mass should increase in proportion to the squared distance). This problem was considered for the first time by Lagrange who reduced it to the quadratures. The qualitative analysis of the Lagrange problem was carried out by Beletskii [6] for the plane case. In space flights it is important to take into account both the homogeneous field (under the action of which a spacecraft gets a constant acceleration), and the field generated by the gravitational center — the planet.

There arises the question of whether it is possible to extend the properties of dynamical systems to the case of a curved space. In particular, it is of interest to study how the curvature affects the integrability of dynamical systems. It is natural to consider the cases of spaces of constant (positive or negative) curvature.

Not so many works are devoted to the study of dynamical systems in spaces of constant curvature. The problem of studying the dynamics in spaces of constant curvature was posed for the first time by Lobachevsky, who studied the generalizations of the gravitation law for the space of constant negative curvature.

The statement of problems of dynamics in spaces of constant curvature is often rather nontrivial. One of the main question is the description of the potential which is generated by a gravitational center. There exist several approaches to the generalization of the classical problems for curved spaces. Moreover, the systems under consideration in celestial mechanics have singularities and, formally speaking, are not integrable in the sense of Liouville (the vector fields generated by the Hamiltonians are not complete). In the work a certain regularization of systems is described; after this regularization the vector fields become complete and smooth.

It should be noted that the number of integrable cases is extremely small, and most of them bear the names of their discoverers. We present to the reader the integrable problems of celestial mechanics on a sphere and in the Lobachevsky space. The problems under consideration are natural generalizations of the classical flat problems of celestial mechanics. However, the integrability of the analogous problems on a sphere and in the Lobachevsky space was noted not long ago [29, 40, 42]. So, the topological properties of these problems are not enough studied yet. In this work a number of interesting results have been obtained. In particular, some of the constructed topological invariants did not appear in integrable cases investigated by many authors earlier. The topology of the isoenergy surfaces is also strongly different from what authors investigated earlier. In this work some new topological effects in the problems of dynamics on the spaces of constant curvature have been discovered. At present time the interest in problems of such type is growing.

This book consists of 5 chapters. In Chapter 1, the basic concepts, definitions and theorems devoted to integrability and the qualitative analysis of dynamical systems, the topological properties of integrable Hamiltonian systems, and the description of their topological invariants and bifurcations are given.

In Chapter 2, the generalization of the Kepler problem on spaces of constant curvature is presented. In this chapter the known results of investigation of the problem on the motion of a particle in a central field in Euclidean space, on a sphere, and in the Lobachevsky space are presented.

In Chapter 3, the results of investigation of the two-center problem in Euclidean space obtained earlier are briefly analysed. The problem on the motion of a point on a sphere (with the standard metrics of constant positive curvature) is studied. The reduction of the problem under consideration to the case of two degrees of freedom was carried out, the integrability is proved, and the integrals of motion are written down. The two centers problem is completely integrable in the sense of Liouville. The reduction to quadratures can be made by the standard method of separation of variables. But it turns out that this problem has very nontrivial topological properties. On the base of the qualitative methods the bifurcation and topological analysis of the system under consideration is carried out, the Fomenko-Ziecshang invariants, which completely describe the topology of Liouville foliations of isoenergy 3-manifolds Q^3, are constructed. All kinds of motion (regular motions and limit motions corresponding to bifurcations of Liouville tori) on the configuration space are described. The connection between Fomenko–Zieschang invariants (marked molecules) and different types of motion are investigated.

In Chapters 4 and 5, the results of studying the integrable problems of celestial mechanics in the Lobachevsky space are set out.

In Chapter 4, the two-center problem in the Lobachevsky space is studied. The problem is reduced to the case of two degrees of freedom, the full integrability is proved, and the integrals of motion are written down. The bifurcation set in the plane of integrals of motion is constructed, and the classification of the domains of possible motion is carried out.

In physics and mechanics, the following problem often arises. Let two integrable Hamiltonian systems be given on symplectic manifolds. It is necessary to know whether they are equivalent in the topological (or Liouville) sense. (Recall that two integrable Hamiltonian systems v_1 and v_2 on symplectic manifolds M_1^4 and M_2^4 (resp. on isoenergy surfaces Q_1^3 and Q_2^3) are called *Liouville equivalent* if there exists a diffeomorphism $M_1^4 \to M_2^4$ (resp. $Q_1^3 \to Q_2^3$) transforming the Liouville foliation of the first system to that of the second one.) In most cases, there exists one unique method to solve this problem. It consists of calculating the corresponding Fomenko–Ziecshang invariants. In this chapter, the theorem on the Liouville equivalence of the two-center problem under the condition that a mass point moves on the upper hemisphere at all time of a motion, on a pseudosphere under the condition $h < -(\gamma_1 + \gamma_2)/R$ and on a plane under the condition $h < 0$, i.e., in the case of bounded motion, is proved.

The connection between integrable systems of celestial mechanics on spaces of constant curvature is investigated as well. It is shown that those problems are transformed one to another as the curvature varies. In particular, it is proved that the Kepler problem and the two centers problem on spaces of nonzero constant curvature λ turn to the corresponding classical problems on the plane.

In Chapter 5, the generalization of the Lagrange problem to the case of the Lobachevsky space is studied. An analog of a constant homogeneous field is obtained. The pseudospheroparabolic coordinates which arise in the process of the passage to the limit are described. Relative to the new coordinates the Hamiltonian has the Liouville form, and, therefore, the full integrability of this problem is demonstrated. The topological and bifurcation analysis is also carried out.

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