G-Convergence and Homogenization of Nonlinear Partial Differential Operators

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Preface

Various applications of the homogenization theory of partial differential equations resulted in the further development of this branch of mathematics, attracting an increasing interest of both mathematicians and experts in other fields. In general, the theory deals with the following:

Let A_k be a sequence of differential operators, linear or nonlinear. We want to examine the asymptotic behaviour of solutions u_k to the equation $Au_k = f$, as $k \to \infty$, provided coefficients of A_k contain rapid oscillations. This is the case, e.g. when the coefficients are of the form $a(\varepsilon_k^{-1}x)$, where the function $a(y)$ is periodic and $\varepsilon_k \to 0$ as $k \to \infty$. Of course, many other kinds of oscillation, like almost periodic or random homogeneous, are of interest as well. It seems a good idea to find a differential operator A such that $u_k \to u$, where u is a solution of the limit equation $Au = f$. Such a limit operator is usually called the *homogenized* operator for the sequence A_k . Sometimes, the term *"averaged"* is used instead of "homogenized".

Let us look more closely what kind of convergence one can expect for u_k . Usually, we have some *a priori* bound for the solutions. However, due to the rapid oscillations of the coefficients, such a bound may be uniform with respect to k in the corresponding energy norm only. Therefore, we may have convergence of solutions only in the weak topology of the energy space. This leads to the notion of *G-convergence* of *abstract* operators in such a way that the homogenized operator A is exactly the G-limit of A_k .

However, while the notion of G-convergence seems to be natural, it is not sufficient for our purpose. Indeed, if A_k is a differential operator, then so it must be for the homogenized operator A. At the same time, the G-limit is defined, in general, only as an abstract operator. To overcome this difficulty it is natural to use an other kind of convergence which is specific for differential operators, namely, *strong G-convergence* (in French literature, the name *H-convergence* is normally used).

Thus, the present book is devoted to strong G-convergence of nonlinear divergence form elliptic and parabolic operators, and applications to homogenization problems proper for periodic, almost periodic, and random homogeneous operators of such kind.

Nevertheless, we start, in Chapter 1, with a discussion of G-convergence for abstract operators, as this theory provides useful tools for the rest of the monograph. Moreover, examination of the situation on this abstract level clarifies some basic ideas. Many results presented here are more or less well-known to experts, but they are scattered in various papers, frequently in an implicit form. It should be pointed out that, beside more or less standard situation, we consider here the case of abstract parabolic operators, which is less familiar.

The core of the book is Chapter 2, in which we study in detail strong G-convergence of nonlinear elliptic operators. We consider both the case of *monotone multivalued,* and *pseudomonotone single-valued* operators. These are treated separately in order to present different approaches. Beside general properties, like strong G-compactness, localization property, and convergence of arbitrary solutions, being essentially common for both cases under consideration, we discuss, in Section 2.4, some additional results which seem to be specific for the single-valued case only. Interesting in themselves these last results provide useful tools for the study of the almost periodic homogenization problem.

Next, in Chapter 3, we discuss nonlinear elliptic homogenization problems. First, we study the case of random homogeneous operators, both single-valued and multivalued. The results we obtain have a *statistical* character, i.e. homogenization takes place for almost all realizations (almost surely). Nevertheless, it is sufficient to get individual homogenization theorems for periodic operators, as an immediate consequence. Then, using the statistical homogenization theorem, general results on strong G-convergence, and the Bohr compactification, we derive an *individual* homogenization theorem for single-valued *almost periodic* operators. Notice that it is unclear how to extend the last result to the case of multivalued operators. Moreover, it is not even known what "almost periodic multivalued operator"means.

Chapter 4 deals with strong G-convergence and homogenization of nonlinear parabolic operators. Here we restrict our study only to the case of single-valued operators. Conceptually, our presentation here is similar to that of Chapter 2 for single-valued elliptic operators. Therefore, we sketch the proofs indicating only main differences. As for homogenization, we consider it only in the periodic setting, but for the whole range of the ratio of time and space scales. In addition, we discuss a class of filtration equations.

We supplement the main body of the monograph by two Appendices. In the first one a version of the homogenization theorem for difference schemes is outlined, while in the second we list some open problems.

Even restricting our work to the subject just described, no attempt has been made to give an exhaustive account of the field or a complete survey of the literature. For additional information we refer to the monographs [40, 47, 113, 164]. We recommend especially the book [164], in which many interesting problems are discussed including homogenization of nonlinear variational problems, and [113] containing a clear and detailed exposition of Γ-convergence. For further results in this direction see, also, papers of A. Braides, G. Dal Maso, R. De Arcangelis, and others, listed in the bibliography. On the other hand, it must be pointed out that the present volume has hardly any overlap with the books cited above.

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Notations

$\mathbf{Z}$	the integers
$\mathbf{N}$	the positive integers
$\mathbf{R}$	the real numbers
$\mathbf{C}$	the complex numbers
p'	the dual exponent, $\frac{1}{p}+\frac{1}{p'}=1$, $p\in[1,\infty]$
$\mathrm{L}^p(Q)$	the usual Lebesgue space on Q with the exponent $p\in[1,\infty]$
$\mathrm{L}^p_{loc}(Q)$	the space of functions which are locally in $\mathrm{L}^p(Q)$
$C_0^\infty(Q)$	the space of infinitely differentiable compactly supported functions
$W^{1,p}(Q)$	the usual Sobolev space of functions in $\mathrm{L}^p(Q)$ whose first derivatives are in $\mathrm{L}^p(Q)$
$W_0^{1,p}(Q)$	the closure of $C_0^\infty(Q)$ in $W^{1,p}(Q)$
$W_{loc}^{1,p}(Q)$	the space of functions which are locally in $W^{1,p}(Q)$
$W^{-1,p'}(Q)$	the dual space to $W_0^{1,p}(Q)$
$H^1(Q)$	$W^{1,2}(Q)$
$H_0^1(Q)$	$W_0^{1,2}(Q)$
$H^{-1}(Q)$	$W^{-1,2}(Q)$
$\mathrm{supp}\, u$	the support of u
$\mathrm{cl}(X)$, $\overline{X}$	the closure of X
$\mathrm{gr}(A)$	the graph of a (multivalued) map A
∇	the gradient operator
∂_t	the time derivative
$\xrightarrow{G}$	G-convergence
$\xRightarrow{G}$	strong G-convergence
I	identity map
$\Vert\cdot\Vert_p$, $\Vert\cdot\Vert_{p,Q}$	the norm in $\mathrm{L}^p(Q)$
$\lvert K\rvert$	the Lebesgue measure of K.