Semigroups and Their Subsemigroup Lattices

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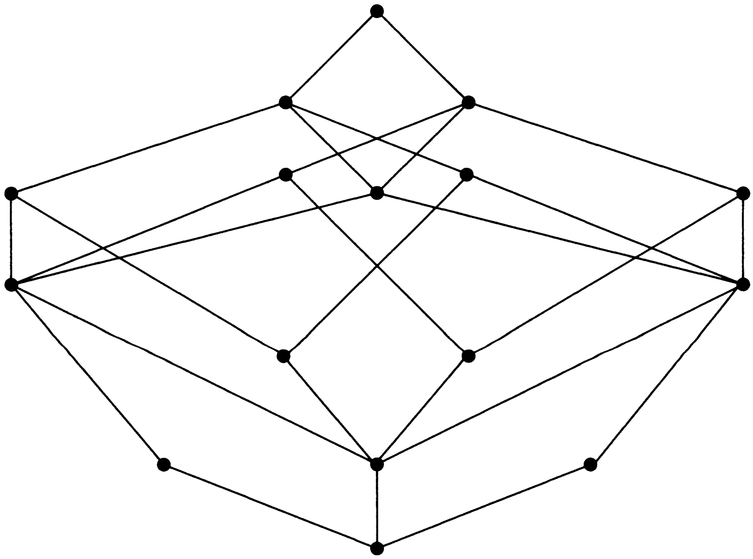
Contents

Part C. Lattice Isomorphisms



Preface

0.1. General remarks. For any algebraic system A, the set Sub$A$ of all subsystems of A partially ordered by inclusion forms a lattice. This is the *subsystem lattice* of A. (In certain cases, such as that of semigroups, in order to have the right always to say that Sub$A$ is a lattice, we have to treat the empty set as a subsystem.) The study of various inter-relationships between systems and their subsystem lattices is a rather large field of investigation developed over many years. This trend was formed first in group theory; basic relevant information up to the early seventies is contained in the book [Suz] and the surveys [K Pek St], [Sad 2], [Ar Sad], there is also a quite recent book [Schm 2]. As another inspiring source, one should point out a branch of mathematics to which the book [Baer] was devoted. One of the key objects of examination in this branch is the subspace lattice of a vector space over a skew field. A more general approach deals with modules and their submodule lattices. Examining subsystem lattices for the case of modules as well as for rings and algebras (both associative and non-associative, in particular, Lie algebras) began more than thirty years ago; there are results on this subject also for lattices, Boolean algebras and some other types of algebraic systems, both concrete and general. A lot of works including several surveys have been published here.

In semigroup theory, investigations concerning subsemigroup lattices have been in progress for more than four decades (the first publications are dated 1951). By now, plentiful and diverse material has been accumulated here. The aim of this monograph is to give a comprehensive and systematized presentation of this material.

It should be noted that results on subsemigroup lattices have been reviewed before in several surveys and monographs. In the book [Lya], two sections contain some early results in this area; one of the parts of the survey [Sad 2] deals with the subsemigroup lattice of a group; in the survey [Ar Sad], two (out of five) sections are devoted to lattice isomorphisms of semigroups of some types, primarily of groups; in the book [Petri 2], one chapter (out of five) is entitled "Lattices of subsemigroups" and concerned mainly with semigroups whose subsemigroup lattices satisfy certain conditions. The survey article [Shev Ov 1] is entirely devoted to such topics; it was the first attempt to review this area comprehensively. The Russian original [Shev Ov 2] of the present monograph was to a considerable degree based on the last survey and gave a revised and, to be sure, more detailed presentation of the material supplemented by some results obtained in the eighties.

In this English version of the monograph, there are certain modifications compared to the Russian original. Firstly, in a number of places we improved the presentation methodically and, furthermore, corrected errors and misprints found in the text. Further, there are several differences in the composition which were induced by a redistribution of the material among sections. As the most considerable effect of modifications of this kind, we note that the main body of Chapter VI consists of results given before without proofs (in a special section "References and supplements" which is present in each chapter); in the Russian original there was not a separate chapter devoted to this topic. On the other hand, we have decided to omit a few proofs and to transfer the corresponding statements to the supplementary sections just mentioned. The last was especially necessary because of another kind of modification: we used the opportunity to include some quite recent results in the book. There is even a whole new chapter (Chapter VIII) which is entirely based on such results. Summarizing, we believe that the present version of our book is definitely better than the previous one (except for, of course, the language: unfortunately, we were far from having a desired stylistic freedom in English).

0.2. Subject-matter lines. When one examines the relationships between a semigroup S and the lattice Sub$S$, the following three main aspects arise[1].

A. *Restrictions on subsemigroup lattices.* A general problem in this aspect is describing the structure of semigroups S for which the lattice Sub$S$ satisfies given lattice-theoretic conditions.

B. *Properties of subsemigroup lattices.* A general problem here is to examine, for a given class of semigroups, the lattices Sub$S$ with S belonging to this class, in particular, to characterize such lattices (within the class of all subsemigroup lattices) and to describe lattices embeddable in the subsemigroup lattices under consideration.

C. *Lattice isomorphisms.* Let S and S' be semigroups. An isomorphism of the lattice Sub$S$ onto the lattice SubS' is called a *lattice isomorphism* of S upon S'. The relation of being lattice isomorphic establishes, so to say, certain kinship between semigroups. It is natural to know a "measure" of this kinship (i.e., to find properties of semigroups which are retained by lattice isomorphisms) and to discover semigroups which are determined by the subsemigroup lattice. It outlines the problems of this aspect (see Section 30).

These three aspects hardly exhaust the whole variety of problems that can be posed when considering subsemigroup lattices, but they embrace the most fundamental problems and reflect the results of practically all investigations carried out hitherto in this area. These aspects naturally determined a basis for the organization of the material in the monograph. As to more concrete lines of examinations, they are first of all cursorily designated by the titles of chapters, while a rather detailed characterization of the content of each chapter together with due motivation is given in the introductions to the chapters and sometimes in the introductory paragraphs of the sections.

[1]These aspects, which we formulate for semigroups, can in fact pertain to the case when S is an arbitrary algebraic system. Notice, furthermore, that (with properly modified formulations) they may apply to respective examinations when, instead of Sub$S$, some other derivative object associated with an algebraic system S is considered, for example, the automorphism group, the endomorphism monoid etc.

Semigroups of certain types can be treated as *unary* semigroups, i.e. semigroups with an additional unary operation. One of the most famous such types is presented by inverse semigroups. The subsystem lattice of a unary semigroup is the lattice of its unary subsemigroups; in the case of inverse semigroups it means the lattice of inverse subsemigroups. Just this case has been to a large extent the object of investigations from the viewpoint in question. The relevant material was reviewed in [Shev Ov 1], [Shev Ov 2] and [Jon 8]; Chapters V, VI and XIV of this book give a complete survey of the results obtained here.

0.3. Structure of the monograph. The contents give a clear general presentation of this structure. Parts A, B, C just correspond to the main aspects indicated above. Each chapter has, besides the numbered sections (the main body of the text of the monograph), a further section "References and supplements". There we indicate the sources of the results and, as we have already mentioned, collect additional results reviewed without proofs. Necessary preliminary facts on semigroups in general (Section 1), inverse semigroups (Section 15) and lattices (Subsection 27.1) are given without proofs as well. Besides, we give without proofs some known facts on groups which are formulated in the relevant places of the text. Each chapter ends with a list of exercises. The assertions we propose to prove in such lists are of different character and include, in particular, some auxiliary facts which are used in the main body of the text. There are in reality many more exercises than in these lists because, as is often done, it is proposed that some easy proofs in the text are carried out by the reader; it concerns, in particular, the proofs of the statements named as observations.

Somewhere we mark unsolved problems and open questions. For the interested reader, we give, at the end of the book, the list of subsections where they are posed. The authorship of results, problems and questions that appear in the main body of the text is generally indicated in the first subsections of the sections "References and supplements". We have modified the original proofs of many results presented in the monograph; as a rule, we do not mention such cases in the text.

The system of (two-index and sometimes three-index) numeration of subsections is fairly obvious; we remark only that, for the sections "References and supplements", the first index means the number of the corresponding chapter written in Roman numerals. The system of reference within the book is also clear; one need only take into account that, referring to an exercise outside [within] the chapter containing this exercise, the number of this chapter is [not] indicated.

In conclusion, we would like to note that during our work both on the Russian and on the English texts of the monograph we discussed various points with a number of colleagues, and we are thankful to them for helpful remarks. Our gratitude is due to the Ural University Press for the publication of the Russian original of this book as well as to the Kluwer Academic Publishers, who kindly suggested the preparation of the present English translation and thereby facilitated the appearance of a revised and expanded version of our monograph.

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