The Theory of Classes of Groups

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Preface

One of the characteristics of modern algebra is the development of new tools and concepts for exploring classes of algebraic systems, whereas the research on individual algebraic systems (e.g., groups, rings, Lie algebras, etc.) continues along traditional lines. The early work on classes of algebras was concerned with showing that one class $\mathcal{X}$ of algebraic systems is actually contained in another class $\mathcal{F}$. Modern research into the theory of classes was initiated in the 1930's by Birkhoff's work [1] on general varieties of algebras, and Neumann's work [1] on varieties of groups. A.I.Mal'cev made fundamental contributions to this modern development.In his reports [1, 3] of 1963 and 1966 to The Fourth All-Union Mathematics Conference and to another international mathematics congress, striking theories of classes of algebraic systems were presented. These were later included in his book [5].

International interest in the theory of formations of finite groups was aroused, and rapidly heated up, during this time, thanks to the work of Gaschütz [8] in 1963, and the work of Carter and Hawkes [1] in 1967. The major topics considered were saturated formations, Fitting classes, and Schunck classes.

A class of groups is called a *formation* if it is closed with respect to homomorphic images and subdirect products. A formation is called *saturated* provided that $G \in \mathcal{F}$ whenever $G/\Phi(G) \in \mathcal{F}$. The theory of saturated formations is derived from the theory of finite soluble groups (the latter enjoyed many remarkable achievements in the early 60's). We will cite some of them as follows.

Hall proved in [1] that if G is a finite soluble group, then there exists, for any non-empty set π of prime numbers, a unique conjugacy class for a π-subgroup H in G, where $(|G:H|, |H|)=1$. Such a subgroup is called a Hall π-subgroup of G. In addition, if T is an arbitrary π-subgroup of G, then there exists a certain $x \in G$ such that $T \subseteq H^x$. This theorem on soluble groups is an impressive result that is analogous to Sylow's theorem. Apart from this, Hall introduced the concept of Sylow basis (cf. Hall [3]). It is worth mentioning that Carter made important contributions to the theory of finite soluble groups. He proved in his paper [2] that there exists a self-normalizing nilpotent subgroup H in a finite soluble group G and that any two subgroups of this kind are conjugate in G. Such a subgroup is known as Carter subgroup.

Gaschütz [8] introduced the following concepts. Let $\mathcal{F}$ be a class of groups and G be a finite group. A subgroup H of G is called an *$\mathcal{F}$-covering subgroup* if the conditions $H \in \mathcal{F}$, $H \leq T \leq G$ and $T/T_0 \in \mathcal{F}$ always imply that $T = T_0H$. At the same time, he proved the following theorem: If $\mathcal{F}$ is a local formation, then there exists a unique conjugacy class of $\mathcal{F}$-covering subgroups in each finite soluble group.

It is known that if $\mathcal{F} = \mathcal{S}_\pi$ (the class of all soluble π-groups), then an $\mathcal{F}$-covering subgroup of a finite soluble group is a Hall π-subgroup of this group, and that if $\mathcal{F} = \mathcal{N}$ (the the class of all nilpotent groups), then the set of all $\mathcal{F}$-covering subgroups of a finite soluble group G turns out to be the set of all Carter subgroups of this group. Therefore Gaschütz's theory puts Hall subgroups and Carter subgroups on an equal footing. Through his method, a series of new subgroups could be produced. For instance, we could infer from the conclusion in Gaschütz [8] that each finite soluble group G has a unique conjugacy class of a supersoluble subgroup H where H has the following properties: if $H \subseteq M \subseteq T$, T is a subgroup of G, and M is a maximal subgroup of T, then $|T : M|$ is not a prime number. Such a subgroup H is called a Gaschütz subgroup of G.

One can ask: for each class $\mathcal{F}$ of groups, does each finite soluble group have an $\mathcal{F}$-covering subgroup? The answer to this question was given by Schunck [1]. Recall that a finite soluble group G is called *primitive* if there exists a maximal subgroup M in G such that $M_G = 1$, and that a class $\mathcal{F}$ of groups is said to be *primitively closed* in the class $\mathcal{S}$ of all finite soluble groups if $\mathcal{F} \subseteq \mathcal{S}$ and if $G \in \mathcal{F}$ whenever all primitive factor groups of a group G belong to $\mathcal{F}$. H. Schunck [1] proved the following important theorem: Let $\mathcal{F} \subseteq \mathcal{S}$. Then there is an $\mathcal{F}$-covering subgoup in every solube group if and only if $\mathcal{F}$ is primitively closed in $\mathcal{S}$ and $\mathcal{F}$ is closed with respect to homomorphic images. A class of groups which meets the conditions of the above theorem is called a *Schunck class*. We shall see that an arbitrary soluble local formation is a Schunk class, but the converse is not always true.

The research into $\mathcal{F}$-covering subgroups leads us to the important concept of $\mathcal{F}$-projector. A subgroup H of a group G is called an $\mathcal{F}$-projector of G provided that, for an arbitrary epimorphism $\varphi : G \to \overline{G}$, the image H^φ of H is a maximal $\mathcal{F}$-subgroup of $\overline{G}$. It is easy to see that, if $\mathcal{F}$ is a formation, then an arbitrary $\mathcal{F}$-covering subgroup of G is an $\mathcal{F}$-projector of G, but the converse is not always true. (The two concepts are equivalent to each other in the class of all finite soluble groups.)

The concept of Fitting class and its related theory appear as a dual of the theory of formations. A class $\mathcal{F}$ of groups is called a *Fitting class* if $\mathcal{F}$ is closed with respect to subnormal subgroups and if the conditions $G = AB$, A, $B \lhd G$ and A, $B \in \mathcal{F}$ always imply $G \in \mathcal{F}$. As a dual of the $\mathcal{F}$-projector in the theory of formations, the concept of $\mathcal{F}$-injector is correspondingly introduced into the theory of Fitting classes. Fischer, Gaschütz and Hartley[1] proved that if $\mathcal{F} \subseteq S$ then there exists a unique conjugacy class of $\mathcal{F}$-injectors in each finite soluble group if and only if $\mathcal{F}$ is a Fitting class. This theorem, together with Gaschütz's theorem on $\mathcal{F}$-covering subgoups, forcefully pushed forward the development of the theory of finite soluble groups and the theory of group classes.

The research on the theory of classes of finite groups started with finite soluble groups in Gaschütz[8] and Carter and Hawkes [1]. However the methodology is subsequently used in studying non-soluble groups, infinite groups, Lie algebras and general algebra.

In the beginning years of the development of the study of classes of finite groups, a great many impressive achievements were made as a result of the efforts of many mathematicians, including W. Gaschütz, R. Baer, B. Huppert, K. Doerk, R. Carter, O. Kegel, T. Hawkes, P. Schmidt, L.A. Shemetkov, H. Neumann, R. Bryant, and others. Most of these achievements have been presented in the publications: Huppert [6], Gaschütz [15], Shemetkov [10], Shemetkov and Skiba [1], and Doerk, Hawkes [4]. In the past ten years or more, especially since the publication of *Formations of Finite groups* by Shemetkov [10], the theory of class of groups has been tremendously developed.

In the initial stage of the development of the theory of classes of groups, such concepts as formation, Fitting class and Schunck class played only a secondary role in the research on the structure of finite groups. But as time went on, people realized that almost all the subjects in the study of groups are related to these classes. Consequently more and more mathematicians showed interest in the research on formations, Fitting classes, Schunck classes, etc. Many of my friends and research fellows now find that it is necessary to have a book which is not very long but good enough to demonstrate, comprehensively, the theory of classes of groups, including its research subjects, major research achievements and research directions, etc. For this purpose I have written this book, and hopefully it will also enable those who are interested in classes of groups to quickly gain access to the frontiers of research in this field.

The first chapter of this book gives a brief introduction to the fundamental concepts of group theory—here we collect almost all the information that the rest of this book requires about the theory of groups. We have two goals in this chapter. First, the detailed information in this chapter makes it possible for the rest of the book not to rely too much on complicated references. This should be most helpful to the reader because it may not be very convenient to consult the related documents as they are not always available or written in other languages. Second, we try our best to make this book accessible to the reader. Although this chapter is based on many books, the proofs of most of the theorems in this chapter are simple and brief.

The second chapter is devoted to the classical portion of the theory of classes of groups consisting mainly of $\mathcal{F}$-covering subgroups, $\mathcal{F}$-projectors, $\mathcal{F}$-injectors and $\mathcal{F}$-normalizers. Most of the theorems in the chapter do not require soluble conditions (at least not for some parts).The second chapter shows the importance of local formations, Schunck classes and Fitting classes in the study of non-simple groups.

The third chapter continues with the research into the formation structures of groups that started in the previous chapter. All groups throughout the chapter are finite groups, and the majority of the research work and results are my own. Section 3.1 describes some methods of constructing local formations, one of which is closely related to the concept of formation function. Let $F(\mathcal{G})$ be the set of all formations of finite groups, and let P be the set of all prime numbers. Then a map f from P to $F(\mathcal{G})$ is said to be a formation function. $LF(f)$ denotes the class of all groups G in which all chief factors are f-central, i.e., $G/C_G(H/K) \in f(p)$ for all chief factors H/K of G, and for all primes p dividing $|H/K|$. Section 3.1 proves that for any formation function f, the class $LF(f)$ is a nonempty local formation. We also prove that the properties of a formation $LF(f)$ are related to the choice of formation function f. Section 3.2 discusses the stability of formations. Section 3.3 focuses on conditions under which the $\mathcal{F}$-coradical $G^{\mathcal{F}}$ of G has a complement in G.

Let $\mathcal{F}$ be a class of groups. A group G is called a minimal non-$\mathcal{F}$-group if $G \notin \mathcal{F}$, but all proper subgroups of G belong to $\mathcal{F}$. When $\mathcal{F}$ is the class of all abelian groups (all nilpotent groups), we obtain the concept of minimal non-abelian group (the concept of Schmidt group). In Section 3.4 we are concerned with the research in the general theory of minimal non-$\mathcal{F}$-groups (where $\mathcal{F}$ is a certain formation). Special attention is also given to the description of Schmidt groups. Section 3.5 describes the local formation $\mathcal{F}$ such that each minimal non-$\mathcal{F}$-group is either a Schmidt group or a group of prime order. Such a formation is called a Shemetkov formation.

Sections 3.6 – 3.10 deal with groups with given properties of subgroups and introduce some of the directions of research and achievements in the theory of formations in recent years.

In Section 3.9, we describe the conditions under which a group belongs to a certain given local formation.

Chapter 4 consists of two parts. The first part describes the correlation between the algebra of variety of groups and the algebra of formations. The second part depicts local formations in which subformations have given algebraic properties. Like the previous chapters, this chapter also covers a series of recent results, some of which are my own.

Chapter 5 has a subsidiary charater. For the convenience of the reader, we have collected the topics from general algebra needed in this book (e.g., the theories of sets, lattices and modules).

Consecutive numbers are given to all the concepts and results in the book.A few of the results are not proved in the book (although many of them are given hints) since they are obvious and simple. Of course the reader may do the job by himself. At the end of each chapter, a special section is devoted to notes or supplementary information including the related historical background knowledge and problems that await to be solved. In this book 29 open problems are raised, some of which are world-famous.

I want to express my appreciation to many who supported my efforts in the course of the writing and publication of this book. I am greatly indebted to Professor Wang E. Fang, Research Fellow Li Fuan and Mr. Liu Jiashan who offered me generous help and valuable suggestions. Special thanks should also be given to Professor A. N. Skiba who provided me with important information regarding some achievements of the Gomel School of Group Theory, and who also made valuable comments on my manuscript.

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