FORMAL METHODS

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RECENT DEVELOPMENT OF SYMBOLIC LOGIC,

SIGNIFICS, SOCIOLOGY OF LANGUAGE,

SOCIOLOGY OF SCIENCE AND OF KNOWLEDGE,

STATISTICS OF LANGUAGE

AND RELATED FIELDS

EVERT W. BETH

# FORMAL METHODS

AN INTRODUCTION TO SYMBOLIC LOGIC

AND TO THE STUDY OF EFFECTIVE OPERATIONS

IN ARITHMETIC AND LOGIC

D. REIDEL PUBLISHING COMPANY / DORDRECHT-HOLLAND

*TO THE MEMORY OF MY MOTHER*

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# PREFACE

Many philosophers have considered logical reasoning as an inborn ability of mankind and as a distinctive feature in the human mind; but we all know that the distribution of this capacity, or at any rate its development, is very unequal. Few people are able to set up a cogent argument; others are at least able to follow a logical argument and even to detect logical fallacies. Nevertheless, even among educated persons there are many who do not even attain this relatively modest level of development.

According to my personal observations, lack of logical ability may be due to various circumstances. In the first place, I mention lack of general intelligence, insufficient power of concentration, and absence of formal education. Secondly, however, I have noticed that many people are unable, or sometimes rather unwilling, to argue *ex hypothesi*; such persons cannot, or will not, start from premisses which they know or believe to be false or even from premisses whose truth is not, in their opinion, sufficiently warranted. Or, if they agree to start from such premisses, they sooner or later stray away from the argument into attempts first to settle the truth or falsehood of the premisses. Presumably this attitude results either from lack of imagination or from undue moral rectitude.

On the other hand, proficiency in logical reasoning is not in itself a guarantee for a clear theoretic insight into the principles and foundations of logic. Skill in logical argumentation is the result of congenital ability combined with practice; theoretic insight, however, can only arise from reflection and analysis.

The purpose of this little book is not to remedy the above-mentioned wide-spread lack of logical ability. I very much doubt the possibility of a cure for lack of intelligence, power of concentration, or imagination. Lack of formal education can, of course, be remedied, but hardly by the study of logic alone. Perhaps this study can be helpful in overcoming the obstacles created by undue moral rectitude.

My main purpose in writing this book has been to explain the principles, foundations, and methods of logic in accordance with contemporary theoretic insight. I have attempted to present this rather subtle subject as simply as possible, but nevertheless I had to presuppose a certain amount of logical skill—about so much as will normally result from the study of elementary mathematics. No previous study of logical theory is required.

For various reasons, I have included a summary discussion of the formalization of arithmetic. But I have neither explained the circumstances which prompted modern developments in logical theory nor the philosophical implications of these developments. For these subjects I may refer to the companion volume on *Mathematical Thought*.

Although I have done my best to maintain a reasonable level of rigour, I have avoided pedantry, real or apparent, in this as well as in other respects. In some cases, I have deliberately and tacitly skipped certain explanations and even proofs, as they might divert the reader's attention from the main course of the argument. If possible, such omissions are made up for as soon as there is an opportunity to return to the subject. But sometimes it seemed better to defer the discussion to a special Section in the *Appendix*.

For those readers who are not professional logicians, the following remarks may be helpful. A book of this kind cannot be read quickly but must be studied carefully. If an isolated, relatively brief, passage presents serious difficulties, it may nevertheless be wise to go ahead and to return later on. If, however, at a certain point various difficulties turn up, this usually means that part of the preceding material has not been rightly understood; it is then advisable to start once more, say, at the beginning of the Section or Chapter.

No exercises are offered, but the reader will have no trouble in finding material by which to test his understanding. For instance, he should draw up deductive and semantic tableaux for the theses which are proved in Section 4. Conversely, he should construct formulas which are logical identities and then try to prove these formulas to be theses on the basis of the axioms stated in Sections 4, 7, and 10.

As to the Bibliography, it is not meant to encourage consulting other publications before the present book, or at least a substantial part of it, has been carefully studied. There still exist many differences in terminology, notation, and method of exposition which, to a certain extent, are motivated by underlying differences of opinion as to the purpose, the principles, and the proper method of logical theory. Unfortunately these differences make it difficult for a beginner to study several books at the same time. Of course, after completing his initiation the reader should try to find his way in the vast literature on the subject.

One final remark is rather meant for professional readers of this book. For classical elementary logic, and for the various logical systems more or less closely connected with it, at least three different methods of deduction are known today and are more or less currently applied in research: Hilbert-type deduction, Gentzen's natural deduction, and Gentzen's calculus of sequents. Logicians often tend to quarrel about the respective merits of these methods. In my opinion, discussions regarding this point are entirely out of place. It seems to me that they only continue because the disputants fail to grasp the fundamental unity of the three methods. In point of fact the three methods must rather be considered as different presentations of one and the same method.

It seems to me essential that a student of logic from the very beginning of his studies be taught all three methods and be made aware of the close connections which exist between them. Moreover, he should become acquainted both with the semantic and with the purely formal approach to the notions, the problems, and the results of logical theory. A dogmatic attitude with respect to the different aspects of logic will easily result if the elements of logic are taught in a narrow spirit.

Furthermore, if due attention is given to the aforementioned different aspects of logical theory, the subject becomes both more interesting and easier. Each one-sided approach leaves part of the material more or less in the dark. For instance, if the semantic background of the formal apparatus is not sufficiently explained, the student will fail to grasp the purpose of the proofs of completeness and thus many of the most important results of research in logic and foundations cannot be adequately understood. It should not be forgotten that later on it is extremely difficult to overcome the bad effects of a narrow-minded initiation.

I wish to express my indebtedness to Professor Hugh Leblanc of Bryn Mawr College who in 1957, by submitting the problem discussed in Section 40, *sub* (1), made me aware of the fact that certain aspects of the method of semantic tableaux deserved renewed attention. His remarks, together with other considerations, led to the introduction and discussion of deductive tableaux. Only after taking this step was I able to take full

advantage of the ideas and results of the authors quoted in Section 6. I have also to thank my friends, Professor B. H. Kazemier and Dr. D. Vuysje, who by their invitation made it possible for me to present the results of my analysis in the form of a book. Miss E. M. Barth, Dr. K. L. de Bouvère, Mr. Horace S. Glover, Mr. J. J. F. Nieland, and Mr. S. C. van Westrhenen have studied the manuscript in various stages of development, and I have to thank them for many improvements both in form and in content. In addition, Miss Barth and Mr. Glover have given considerable help in proof-reading and so has also Mrs. Gay Honeywood.

E. W. BETH

Amsterdam, October, 1961.

# REMARKS ON TERMINOLOGY AND NOTATION

The symbols and formulas appearing in the printed text of stipulations (F1)-(F3) in Section 1 (*cf.* Sections 7, 8, and 37) are best understood as names or descriptions of the symbols and formulas of logic, or as variables ranging over the sets of all these symbols and formulas. Likewise, the first and fourth formulas appearing in stipulation ($F2^b$) in Section 8 are to be understood, respectively, as variables ranging over the relevant subsets of the sets of all formulas and of all expressions of logic. The symbols and formulas of logic are only discussed, they are never actually displayed.

We also discuss finite sets $K$, $L$, $(K', U)$, $(U_1, U_2, \ldots, U_m)$, ... of logical formulas. A sequent $K/L$ is an ordered couple of such sets (*cf.* Section 33) and *not*, as in Gentzen, a logical formula in its own right. Accordingly, no specific calculus of sequents is developed, but sequents and (deductive and semantic) tableaux are used as tools in the metamathematical analysis of certain logical systems. Practically, however, the study of deductive and semantic tableaux provides a convenient alternative to the metamathematical investigation of Gentzen's calculi LJ and LK.

The symbols appearing in the text of Section 17, *sub* (4), are meant to describe certain numerals which, in turn, are assumed to denote certain natural numbers.

By *semantics* I mean a rigorously deductive treatment of the connections between the logical and mathematical symbols and the objects which they denote. An informal discussion of the same subject is denoted as *hermeneutics*.

In connection with the discussion in the Preface, I may refer to similar remarks on lack of logical ability which are made by J. Castiello, *Geistesformung*, Berlin-Bonn 1934, Ss. 74–76.

The title of Section 19 must be understood in connection with the explanations given in Section 22, *sub* (1).