

**AN OUTLINE OF MATHEMATICAL LOGIC**

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# AN OUTLINE OF MATHEMATICAL LOGIC

*Fundamental Results and Notions  
Explained with All Details*

*by*

ANDRZEJ GRZEGORCZYK

*Warsaw University*



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*TO MY PARENTS*

## PREFACE

Recent years have seen the appearance of many English-language handbooks of logic and numerous monographs on topical discoveries in the foundations of mathematics. These publications on the foundations of mathematics as a whole are rather difficult for the beginners or refer the reader to other handbooks and various piecemeal contributions and also sometimes to largely conceived “mathematical folklore” of unpublished results. As distinct from these, the present book is as easy as possible systematic exposition of the now classical results in the foundations of mathematics. Hence the book may be useful especially for those readers who want to have all the proofs carried out in full and all the concepts explained in detail. In this sense the book is self-contained. The reader’s ability to guess is not assumed, and the author’s ambition was to reduce the use of such words as *evident* and *obvious* in proofs to a minimum. This is why the book, it is believed, may be helpful in teaching or learning the foundation of mathematics in those situations in which the student cannot refer to a parallel lecture on the subject. This is also the reason that I do not insert in the book the last results and the most modern and fashionable approaches to the subject, which does not enrich the essential knowledge in foundations but can discourage the beginner by their abstract form.

A. G.

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