A First Course on Complex Functions

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Edited by Ronald Brown
University College of North Wales
Bangor

J. de Wet
Balliol College, Oxford

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G. J. O. JAMESON

Lecturer in Mathematics
University of Warwick

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Preface

This book contains a rigorous coverage of those topics (and only those topics) that, in the author's judgement, are suitable for inclusion in a first course on Complex Functions. Roughly speaking, these can be summarized as being the things that can be done with Cauchy's integral formula and the residue theorem. On the theoretical side, this includes the basic core of the theory of differentiable complex functions, a theory which is unsurpassed in Mathematics for its cohesion, elegance and wealth of surprises. On the practical side, it includes the computational applications of the residue theorem. Some prominence is given to the latter, because for the more sceptical student they provide the justification for inventing the complex numbers.

Analytic continuation and Riemann surfaces form an essentially different chapter of Complex Analysis. A proper treatment is far too sophisticated for a first course, and they are therefore excluded.

The aim has been to produce the simplest possible rigorous treatment of the topics discussed. For the programme outlined above, it is quite sufficient to prove Cauchy's integral theorem for paths in star-shaped open sets, so this is done. No form of the Jordan curve theorem is used anywhere in the book.

The results of Complex Analysis are constantly motivated and illustrated by comparison with the real case. This policy is exemplified by the proof of Cauchy's integral theorem. Since similar formulae hold in the two cases for the integral of a derivative, it is enough to show that a differentiable function is itself a derivative. For a real function f, this would be done by showing that it is the derivative of F, where $F(x) = \int_a^x f$. The attempt to do the same in the complex case leads straight to the necessity of proving the Cauchy-Goursat theorem for a triangle.

The prerequisites are elementary Real Analysis and a little familiarity with metric spaces. The required metric space theory is set out on pages 1–7. This summary is strictly limited to results needed elsewhere in the book, and is not in any way intended as an alternative to the various existing texts on metric spaces. Though logically self-contained, this section is designed as a brisk revision rather than a beginner's introduction to metric spaces. Most, if not all, universities now include a course on metric or topological spaces quite early in their undergraduate programme, a policy which removes the necessity of beginning other courses with a lengthy discussion of preliminaries.

Chapter 2 contains the central part of the subject – the theory of functions that are differentiable on an open set. Digressions are kept to a minimum, in order not to detract from the beauty and unity of this classic sequence of results. The computational applications of the residue theorem are left to chapter 3, and all preparatory concepts, including integration, are fully discussed in chapter 1. Winding numbers are defined in order to state the residue theorem, but they are not permitted to interrupt the flow, on the grounds that all the usual applications of the residue theorem use paths that are easily seen to be simple. A proper discussion of winding numbers is deferred to chapter 3.

Chapter 1 may lack the unity and aesthetic appeal of chapter 2, but it would be too superficial to dismiss it as being merely introductory, or as being more typical of Real than Complex Analysis. Power series and the exponential function are really basic to Complex Analysis; it is debatable whether they are basic to Real Analysis. However, many of the results and proofs in chapter 1 have counterparts in the real case, and it should be possible to cover most of the material fairly quickly. Proofs are only omitted when the author is confident that all Real Analysis courses include them. For example, a proof is given of the theorem on differentiation of power series, but not of the rule for differentiating a product.

The section on the exponential and trigonometric functions should be viewed as a unit on its own. The properties of these functions are deduced from the power-series definitions, without assuming any results about the corresponding real functions. This procedure has the advantage of avoiding any logical gaps that might arise if a different definition had been used in a previous Real Analysis course. Omission of the proofs that do not involve complex numbers would result in an incomplete and disjointed account, without actually saving very much space.

We follow Ahlfors in defining paths to be functions defined on real intervals (and not equivalence classes of such functions). Integration is defined by approximating with step functions, since this serves to give some intuitive meaning to the integral, but it is made possible for the reader to opt for the purely formal definition of the integral as an integral of a function from **R** to **C**.

Two slightly esoteric topics that are included both require an acknowledgement. I am indebted to Mr D. H. Fowler for showing me the method of evaluating the probability integral reproduced in 3.1, and to Prof. D. B. A. Epstein for drawing my attention to Palais' beautiful proof of the paucity of finite-dimensional division algebras (2.3, exercise 7). I am also indebted to the Editor of the Chapman and Hall Mathematics Series, Dr R. Brown, for reading the manuscript and making numerous useful suggestions.

Terminology and notation

The set of real numbers is denoted by **R**, and the set of complex numbers by **C**. Functions whose domain and range are both subsets of **R** are called *real functions*, and those whose domain and range are both subsets of **C** are called *complex functions*.

Terms and symbols relating to sets and mappings are used with their usual meaning. We write $f(x)$ for the image of x under the mapping f, and (occasionally) $f(A)$ for the set $\{f(x):x \in A\}$. Set-theoretic difference is denoted by $\backslash$, and composition of functions by $\circ$. The closed real interval $\{x:a \leqslant x \leqslant b\}$ is denoted by $[a,b]$, and the open interval $\{x:a < x < b\}$ by (a,b). Other symbols introduced in the text are listed on pages 143–4.

Results (and only results) are numbered consecutively within sections, so that those in section 1.7 are 1.7.1, 1.7.2, etc. Only the most important ones are dignified with the appellation 'theorem'. Lemmas and corollaries are so designated, but all results not belonging to any of these categories are numbered without verbal qualification.

The derivative of the real function f at a (if it exists) is denoted by $f'(a)$, and the integral of f on $[a, b]$ by $\int_a^b f$. Similar notation is used for complex functions once the corresponding notions have been defined. It is, however, indisputably convenient to be able to use the classical notation d/dx and $\int \ldots dx$ on occasions. Its meaning can be made precise as follows. A function can be denoted either by giving it a name (such as f) or by the 'arrow' convention, e.g. $x \mapsto x^2$. The second method is often useful when dealing with particular functions, although it does not give any way of denoting the value of a function at a particular argument. When it is employed, we agree that the function f' can be denoted by $x \mapsto (d/dx)f(x)$. Here x, at its three occurrences, could be replaced by any other symbol not already used. With this understanding, it is correct to write (for example):

$$\frac{d}{dx}x^2 = 2x, \qquad \frac{d}{dt}t^2 = 2t.$$

In the case of integration, we agree that $\int_a^b f$ may be denoted by $\int_a^b f(x)dx$; again, another symbol could replace x.