The Assay of Spatially Random Material

Mathematics and Its Applications

The Assay of Spatially Random Material

by

Yakov Ben-Haim

Department of Nuclear Engineering,
Technion-Israel Institute of Technology, Haifa, Israel

D. Reidel Publishing Company

A MEMBER OF THE KLUWER ACADEMIC PUBLISHERS GROUP

Dordrecht / Boston / Lancaster / Tokyo

Library of Congress Cataloging in Publication Data

CIP

Ben-Haim, Yakov, 1952-
The assay of spatially random material.

(Mathematics and its applications)
Includes bibliographies and indexes.
1. Materials– Analysis–Statistical methods. I. Title.
II. Series: Mathematics and its applications (D. Reidel Publishing
Company)
QD131.B39 1985 543 85-14221

ISBN-13: 978-94-010-8893-0 e-ISBN-13: 978-94-009-5422-9
DOI: 10.1007/978-94-009-5422-9

Published by D. Reidel Publishing Company
P.O. Box 17, 3300 AA Dordrecht, Holland

Sold and distributed in the U.S.A. and Canada
by Kluwer Academic Publishers,
190 Old Derby Street, Hingham, MA 02043, U.S.A.

In all other countries, sold and distributed
by Kluwer Academic Publishers Group,
P.O. Box 322, 3300 AH Dordrecht, Holland

Softcover reprint of the hardcover 1st edition 1985

לאשתי מרים

TO MIRIAM

TABLE OF CONTENTS

Chapter 4
Probabilistic Interpretation of Measurement

Chapter 5
Probabilistic Design

Chapter 6
Adaptive Assay

EDITOR'S PREFACE

Approach your problems from the right end and begin with the answers. Then one day, perhaps you will find the final question.

'The Hermit Clad in Crane Feathers' in R. van Gulik's *The Chinese Maze Murders*.

It isn't that they can't see the solution. It is that they can't see the problem.

G.K. Chesterton. *The Scandal of Father Brown* 'The point of a Pin'.

Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the "tree" of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related.

Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as "experimental mathematics", "CFD", "completely integrable systems", "chaos, synergetics and large-scale order", which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics. This programme, Mathematics and Its Applications, is devoted to new emerging (sub)disciplines and to such (new) interrelations as exempla gratia:

- a central concept which plays an important role in several different mathematical and/or scientific specialized areas;
- new applications of the results and ideas from one area of scientific endeavour into another;
- influences which the results, problems and concepts of one field of enquiry have and have had on the development of another.

The Mathematics and Its Applications programme tries to make available a careful selection of books which fit the philosophy outlined above. With such books, which are stimulating rather than definitive, intriguing rather than encyclopaedic, we hope to contribute something towards better communication among the practitioners in diversified fields.

Because of the wealth of scholarly research being undertaken in the Soviet Union, Eastern Europe, and Japan, it was decided to devote special attention to the work emanating from these particular regions. Thus it was decided to start three regional series under the umbrella of the main MIA programme.

One characteristic of present day (applied) mathematics is the regular discovery or emergency of new topics to which it can be successfully applied. And the new (from the mathematical point of view) kinds of problems which thus arise, often calling for new ideas of analysis and synthesis. Electrical engineering is one such seemingly inexhaustible source of new interesting problems. Another vast class concerns all kind of problems where knowledge is needed about physically inaccessible spatially distributed structures on the basis of derived signals (such as scattering data or tomographic, i.e. Radon, integrals) to be obtained by the mathematical treatment of the available indirect data. The assay of (radioactive) inhomogeneous materials is one such topic. It generates new problems and calls for new mathematical frameworks, of course not necessarily always immediately of the level of sophistication of more established and older branches of mathematics, but certainly fascinating in the different flavour of the ideas involved and techniques needed. As in many of these new fields, linked directly or indirectly with data processing, the matters of computer algorithms and complexity (as opposed to difficulty) of design play an significant role.

This is a unique book on the design of assay-systems and may well turn out to be a focal point from which one more substantial branch of applied mathematics will flourish. The ingredients, apart from the original problem, include probability, statistics, theory of measurement, control theory and adaptive ideas: a promising mix.

The unreasonable effectiveness of mathematics in science ...

Eugene Wigner

Well, if you know of a better 'ole, go to it.

Bruce Bairnsfather

What is now proved was once only imagined.

William Blake

As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited.

But when these sciences joined company they drew from each other fresh vitality and thenceforward marched on at a rapid pace towards perfection.

Joseph Louis Lagrange.

Bussum, July 1985

Michiel Hazewinkel

PREFACE

The assay of material is performed in a multitude of applications. In many situations the structure of the sample is known. By far the most common morphology, occurring almost invariably in the analytical chemistry of solutions, is the homogeneous sample. Despite the importance of such assays, our attention in this volume will be directed to those assay applications in which the structure of the sample is unknown or imprecisely known to the analyst. Not only the distribution of the various components throughout the spatial confines of the sample may be unknown, but also the external shape and dimensions may be hidden. Since the precise morphology of the sample is unknown, one may image that the sample has been drawn from a collection of possible morphologies. It is convenient to refer to this situation as *spatial randomness*, and to describe the assayed material as being *spatially random*. In the introductory Chapter, and indeed throughout the book, we shall discuss numerous examples of spatially random material. From the outset, however, it is important to recognize two points.

First, incomplete knowledge of the structure of the sample causes major complications in the design of a reliable assay system, and in the interpretation of the measurements obtained therefrom. The difficulty in assaying a sample in the absence of precise knowledge of its structure stems from the following fact: different quantities of the material whose assay is to be determined can be differently distributed in the confines of distinct samples, so as to appear identical when assayed. In order to remove this ambiguity from the assay, it is necessary to employ instrumental designs which are totally irrelevant when the sample morphology is known. Furthermore, when the sample morphology is unknown it is likely to be far from adequate to base measurement-calibrations on an assumption of homogeneity (or on any other fixed sample structure).

The second point, which provides the major motivation for this book, is that a single mathematical framework has been established which underlies the assay of spatially random material. The reader will have no difficulty accepting this statement as it pertains to data interpretation, since it will be sufficient to employ standard tools, of acknowledged generality, from the statistical theories of decision and estimation. The case is less clear with regard to assay-system design, but it would be premature to try to prove the point in the preface. Let it suffice to say that a general concept of design optimality can be formulated and studied for a very broad class of spatially random materials, regardless of whether the specific application is geological, medical, meteorological, hydrodynamical or whatever.

This book has been written with two goals in mind. It is intended to facilitate the application of a body of mathematical concepts to the practical tasks of assay-system design and data-interpretation. A fair portion of the mathematical material, especially that relating to assay-system design, has appeared only in technical journals which are not readily accessible to workers in the wide range of disciplines for which this material is relevant. It is hoped that the large number of examples presented throughout the book will assist the reader in applying, in his own field of endeavor, the ideas which are discussed. In light of this aim the examples are by and large heuristic. Complicated computer calculations are avoided, as these obscure the concepts which must be implemented in the course of design or data interpretation. Wherever possible analytic simplifications are employed so as to enable the reader to follow in detail the method of analysis. However, the techniques discussed are by no means limited to simple applications. Great pains are taken to develop computerizable algorithms by means of which multi-detector systems of extraordinary complexity may be efficiently and rationally designed.

The second aim of the book is to engender further study of the mathematics itself. Consequently, theorems are presented with proof, especially in the Chapters on assay-system design. The concept of spatially random material is quite broad, and many extensions and applications remain unexplored. The final Chapter is devoted to a brief discussion of some directions for further research.

The two aims of the book are to some extent conflicting. The practitioner may view the detailed proofs as extraneous, while the theoretically inclined reader may find the abundance of examples to be distracting. In an attempt to resolve this dilemma, the examples are separated from the mathematical material by appearing in distinct sections. The examples are also fairly self contained. On the other hand, the theoretician should not deprive himself of a good understanding of the problems faced in practice. Similarly, the application-oriented worker is certainly aware that a poor understanding of the concepts underlying a technique may lead to incomplete exploitation or even misuse of the theoretical tools.

Chapter 2 contains the conceptual basis for the design of assay systems. Generalizations and proofs are deferred to Chapter 3. Chapter 4 is devoted to a discussion of the probabilistic interpretation of measurement. In Chapter 5 the statistical tools introduced in Chapter 4 are applied to the topic of assay-system design. Chapter 6 contains an exposition of the technique of adaptive assay, in which the functions of design and interpretation are integrated into a single unit which operates in the course of measurement.

Numerous people have contributed in many ways to the development of this book. Only a few can be mentioned. I am pleased to acknowledge the contributions of Professors Amos Notea and Yitzhak Segal, for it is from their pioneering work on probabilistic nondestructive assay that the present effort has grown. I am deeply indebted to Dr. Natan Shenhav and Professors Ezra Elias and Tsahi Gozani, whose collaboration in various stages of this research was invaluable. Dr. Shenhav's critical reading of a major portion of this book led to the removal of many obscurities.

Yakov Ben-Haim

Haifa
1985

NOTATION

The following symbols are used consistently throughout the book.

SYMBOL	MEANING
X	spatial domain of the sample.
$f(x)$	point-source response function.
$x[h]$	$x_1^{h_1} x_2^{h_2} \cdots x_n^{h_n}$ where h is a vector of non-negative integers.
$r(x)$	source density at point x in the sample.
$F(h)$	point-source response set for the h-moment.
$R(h, u)$	set of all spatial distributions whose h-moment equals u.
$\widetilde{R}(h, u)$	constrained set of allowed spatial distributions whose h-moment equals u.
$C(h, u)$	complete response set for spatial distributions in $R(h, u)$.
$\widetilde{C}(h, u)$	complete response set for spatial distributions in $\widetilde{R}(h, u)$.
$K(Y)$	set of all continuous functions on the set Y.
$E(n)$	n-dimensional real Euclidean vector space.
$\mathrm{E}(A)$	statistical expectation of A.