

Solid Mechanics and Its Applications

Volume 201

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The median level of presentation is the first year graduate student. Some texts are monographs defining the current state of the field; others are accessible to final year undergraduates; but essentially the emphasis is on readability and clarity.

Meinhard Kuna

Finite Elements in Fracture Mechanics

Theory—Numerics—Applications

 Springer

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Preface

Prevention and assessment of fracture and damage processes play an essential role in the development and dimensioning of engineering constructions, components, and facilities in order to ensure their technical safety, durability, and reliability. In the case of failure, mistakes made by engineers in this respect can have catastrophic consequences for the lives of people, the environment, and even for the economy. In many engineering components and materials, defects may exist resulting from manufacturing or operation, which cannot always be avoided. Therefore, the fracture mechanical assessment of crack-like defects is of great importance. In the context of technical surveillance and studies of causes of failure cases, besides materials characterization the analysis of the mechanical loading situation at cracks, notches, and similar defects under in-service conditions is of particular interest.

In the past 50 years, *fracture mechanics* has been developed into an independent interdisciplinary scientific field, which resides between engineering mechanics, materials science, and solid-state physics. Fracture mechanics defines load parameters and criteria in order to quantitatively assess crack behavior in materials and components under static, dynamic, or cyclic loading.

Additionally, numerical methods of applied mechanics are used nowadays for fracture-mechanical stress analysis. The finite element method (FEM) has been established in many areas of engineering as a universal and efficient tool of modern engineering design and stress analysis. Numerous software packages are available, which offer not only standard methods of structural mechanics but also fracture-mechanical options of more recent invention. However, the treatment of crack issues requires particular theoretical precognition and numerical algorithms, much of which has not yet been integrated into the engineer's education and practice to the necessary extent, but has been available mostly to »fracture-mechanical experts« only.

The intention of the present monograph consists in closing this gap. In the introduction, we present the essential theoretical basics of fracture mechanics, whose parameters are to be determined using the FEM. The main part of the book is focused on specific numerical techniques to analyze plane and spatial crack

problems in elastic and plastic materials under all technically relevant loads. Finally, worked samples for the solutions of practical problems will be provided for each area.

This textbook is addressed to graduate students of engineering study courses, especially those in mechanical engineering, civil engineering, vehicle design, materials science, aerospace industry, or computational engineering. It shall provide an introduction into this area of expertise to graduates and postgraduates of these fields and support in their own research activities. Moreover, I consider as a target audience engineers in design and computation departments of many industrial branches and officials in technical controlling institutions, who are confronted with issues of dimensioning, assessment, and supervision of strength and durability of engineering constructions. Furthermore, this textbook should build a bridge for materials scientists and materials engineers to theoretical fracture mechanics in order to use numerical techniques for materials modeling or to analyze materials and components tests using computations. This textbook requires a basic knowledge of continuum mechanics, strength of materials, material theory, and the finite-element method. The essential basics of mechanics of materials are reviewed for convenience in the Appendix.

I was gratified at the positive response by which the scientific community in Germany has appreciated the first edition of this book entitled “Numerische Beanspruchungsanalyse von Rissen—Finite-Elemente in der Bruchmechanik”, edited by Vieweg-Teubner publisher in 2008. Many readers confirmed to me personally that the book was a useful help to understand fracture mechanics concepts and a real assistance in performing their own numerical computations. Meanwhile, the second improved edition appeared in 2010. Therefore, I feel encouraged to offer this book to a wider audience in the English language.

Many persons contributed to the preparation of the book. First of all, my very sincere thanks go to Ms. M. Beer for making all the excellent drawings. Numerous numerical examples were elaborated during the pleasant joint work with my former and current Ph.D.-students or co-workers at the institute. In particular, I would like to express my thanks to Dr. M. Abendroth, Dr. M. Enderlein, Dr. E. Kullig, Th. Leibelt, C. Ludwig, Dr. U. Mühlich, Dr. F. Rabold, Dr. B. N. Rao, Prof. Dr. A. Ricoeur, Dr. A. Rusakov, L. Sommer, and L. Zybell.

I appreciate the fruitful cooperation lasting for many years with my colleagues Prof. Dr. G. Pusch (TU Freiberg) and Prof. Dr. P. Hübner (University Mittweida), from where the reported engineering applications of fracture assessment have emerged. Also, I am indebted to Prof. Dr. M. Fulland (University Zittau) and Dr. I. Scheider (GKSS Geesthacht), who provided me kindly graphical material for additional examples. My thanks go to Prof. Dr. W. Brock for reviewing the German manuscript and giving constructive comments on the scientific presentation of the subject.

In the course of translating and revising the English manuscript, I appreciate the great assistance by A. Kuchle (Chaps. 4–7) E. Beschler, and J. Bergemann.

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Last but not least, I cordially want to thank my wife, Christine Kuna, for her great understanding and infinite patience.

Freiberg, December 2012

Meinhard Kuna

Contents

1	Introduction	1
1.1	Fracture Phenomena in Nature and Engineering	1
1.2	Fracture Mechanics	5
1.3	Computational Methods for Cracks	8
1.4	Basic Literature on Fracture Mechanics	10
2	Classification of Fracture Processes	13
2.1	Macroscopic Manifestations of Fracture	13
2.2	Microscopic Appearances of Fracture	17
2.3	Classification of Fracture Processes	18
	Reference	20
3	Basics of Fracture Mechanics	21
3.1	Model Assumptions	21
3.2	Linear-Elastic Fracture Mechanics	23
3.2.1	Two-Dimensional Crack Problems	23
3.2.2	Eigenfunctions of the Crack Problem	31
3.2.3	Three-Dimensional Crack Problems	35
3.2.4	Stress Intensity Factors: K-Concept	38
3.2.5	Energy Balance During Crack Propagation	42
3.2.6	The J-Integral	51
3.2.7	Cracks in Anisotropic Elastic Bodies	54
3.2.8	Interface Cracks	58
3.2.9	Cracks in Plates and Shells	62
3.2.10	Fracture Mechanical Weight Functions	65
3.2.11	Thermal and Electric Fields	76
3.3	Elastic-Plastic Fracture Mechanics	80
3.3.1	Introduction	80
3.3.2	Small Plastic Zones at the Crack	81
3.3.3	The Dugdale Model	86
3.3.4	Crack Tip Opening Displacement (CTOD)	87
3.3.5	Failure Assessment Diagram (FAD)	89
3.3.6	Crack Tip Fields	91

3.3.7	The J-Integral Concept	101
3.3.8	Ductile Crack Growth.	106
3.4	Fatigue Crack Propagation	113
3.4.1	Constant Amplitude Loading	114
3.4.2	Stress State at the Crack Tip	118
3.4.3	Variable-Amplitude Loading	121
3.4.4	Fracture Criteria at Mixed-Mode Loading	124
3.4.5	Fatigue Crack Growth at Mixed-Mode Loading.	130
3.4.6	Prediction of Crack Path and its Stability	131
3.5	Dynamic Fracture Processes	133
3.5.1	Introduction.	133
3.5.2	Fundamentals of Elastodynamics	134
3.5.3	Dynamic Loading of Stationary Cracks	135
3.5.4	Dynamic Crack Propagation	137
3.5.5	Energy Balance and J-Integrals	142
3.5.6	Fracture Criteria.	145
	References	146
4	Finite Element Method.	153
4.1	Spatial and Temporal Discretization of Boundary Value Problems	153
4.2	Energy Principles of Continuum Mechanics	156
4.2.1	Variation of Displacement Field	156
4.2.2	Variation of Forces	159
4.2.3	Mixed and Hybrid Variational Principles	161
4.2.4	Hamilton's Principle.	165
4.3	Basic Equations of FEM.	166
4.3.1	Constructing Stiffness Matrix for One Element	167
4.3.2	Assembly and Solution of the Total System	169
4.4	Numerical Realization of FEM	171
4.4.1	Selection of Displacement Functions	171
4.4.2	Isoparametric Element Family	171
4.4.3	Numerical Integration of Element Matrices.	175
4.4.4	Numerical Interpolation of Results.	177
4.5	FEM for Non-Linear Boundary Value Problems	180
4.5.1	Basic Equations	180
4.5.2	Material Non-linearity	183
4.5.3	Geometrical Non-linearity.	186
4.6	Explicit FEA for Dynamic Problems	189
4.7	Procedure of a Finite Element Analysis	191
4.7.1	PRE-processing	191
4.7.2	FEM-processing.	191
4.7.3	POST-processing	191
	References	192

5	FE-Techniques for Crack Analysis in Linear-Elastic Structures . . .	193
5.1	Interpreting the Numerical Solution at the Crack Tip	193
5.2	Special Finite Elements at the Crack Tip	197
5.2.1	Development of Crack Tip Elements	197
5.2.2	Modified Isoparametric Displacement Elements.	198
5.2.3	Computing Intensity Factors from Quarter-Point Elements	207
5.3	Hybrid Crack Tip Elements	212
5.3.1	Development of Hybrid Crack Tip Elements	212
5.3.2	2D Crack Tip Elements Based on Mixed Hybrid Model	214
5.3.3	3D Crack Tip Elements Based on Hybrid Stress Model	218
5.4	Method of Global Energy Release Rate	223
5.4.1	Realization Within FEA	223
5.4.2	Method of Virtual Crack Extension	224
5.5	Method of Crack Closure Integral	227
5.5.1	Basic Equations of Local Energy Method	227
5.5.2	Numerical Implementation in FEA 2D	229
5.5.3	Numerical Implementation in FEA 3D	233
5.5.4	Consideration of Crack Face, Volume and Thermal Loading	239
5.6	FE-Computation of J -Contour Integrals	240
5.7	FE-Calculation of Fracture Mechanics Weight Functions	243
5.7.1	Determination by Point Forces	243
5.7.2	Determination of Parametric Influence Functions	245
5.7.3	Derivation from Displacement Fields	247
5.7.4	Application of the J -VCE-Technique	249
5.7.5	Calculation by Means of the Bueckner-Singularity	250
5.8	Examples	251
5.8.1	Tension Sheet with Internal Crack	251
5.8.2	Semi-Elliptical Surface Crack Under Tension	255
	References	258
6	Numerical Calculation of Generalized Energy Balance Integrals	263
6.1	Generalized Energy Balance Integrals	263
6.2	Extension to General Loading Cases	267
6.2.1	Preconditions for Path-Independence	267
6.2.2	Crack Face, Volume and Thermal Loading	268
6.3	Three-Dimensional Variants	270
6.3.1	The 3D-Disk Integral	271
6.3.2	Virtual Crack Propagation in 3D	273

- 6.4 Numerical Calculation as Equivalent Domain Integral 275
 - 6.4.1 Transformation into an Equivalent Domain
Integral 2D 275
 - 6.4.2 Transformation into an Equivalent Domain
Integral 3D 278
 - 6.4.3 Numerical Implementation 279
- 6.5 Consideration of Dynamic Processes 281
- 6.6 Extension to Inhomogeneous Structures 283
- 6.7 Treatment of Mixed-Mode-Crack Problems. 285
 - 6.7.1 Separation into Crack Opening Modes I and II 285
 - 6.7.2 Interaction-Integral-Technique 288
- 6.8 Calculation of *T*-Stresses 291
- 6.9 Examples 294
 - 6.9.1 Internal Crack Under Crack Face Loading 294
 - 6.9.2 Edge Crack Under Thermal Shock. 296
 - 6.9.3 Dynamically Loaded Internal Crack. 298
 - 6.9.4 Crack in a Functionally Graded Material 300
- 6.10 Concluding Assessment of Methods. 302
- References 303

- 7 FE-Techniques for Crack Analysis in Elastic-Plastic Structures. 305**
 - 7.1 Elastic-Plastic Crack Tip Elements. 305
 - 7.2 Determination of Crack Tip Opening Displacements 308
 - 7.3 Calculation of the *J*-Integral and its Meaning 309
 - 7.3.1 Elastic-Plastic Extensions of *J*. 309
 - 7.3.2 Application to Stationary Cracks 314
 - 7.3.3 Application to Moving Cracks. 316
 - 7.4 Examples 317
 - 7.4.1 Compact-Tension Specimen 317
 - 7.4.2 Tensile Plate with Surface Crack. 321
- References 325

- 8 Numerical Simulation of Crack Propagation. 327**
 - 8.1 Nodal Release Technique 328
 - 8.2 Techniques of Element Modification 329
 - 8.2.1 Element Splitting 329
 - 8.2.2 Element Elimination Technique. 331
 - 8.2.3 Adapting Element Stiffness. 332
 - 8.3 Moving Crack Tip Elements 333
 - 8.4 Adaptive Remeshing Strategies 335
 - 8.4.1 Error-Controlled Adaptive Meshing 335
 - 8.4.2 Simulation of Crack Propagation 336

8.5	Cohesive Zone Models	338
8.5.1	Physical Background	338
8.5.2	Numerical Realization	343
8.6	Damage Mechanical Models	345
8.7	Examples of Fatigue Crack Propagation	347
8.7.1	Shear Force Bending Specimen	347
8.7.2	ICE-Wheel Failure	349
8.8	Examples of Ductile Crack Propagation	351
8.8.1	Cohesive Zone Model for CT-Specimen	351
8.8.2	Damage Mechanics for SENB-Specimen	354
	References	358
9	Practical Applications	361
9.1	Fatigue Crack Growth in a Railway Wheel	361
9.1.1	Material Data of Austempered Ductile Iron ADI	361
9.1.2	Finite Element Calculation of the Wheel	362
9.1.3	Specification of Crack Postulates	365
9.1.4	Fracture Mechanical Analysis	366
9.2	Brittle Fracture Assessment of a Container Under Impact Loading	373
9.2.1	FE-Model of the Drop Test	373
9.2.2	Fracture Mechanical Results of the Simulation	376
9.2.3	Application of Submodel Technique	376
9.3	Ductile Fracture of a Weldment in a Gas Pipeline	377
9.3.1	Introduction	377
9.3.2	Fracture Mechanics Assessment Concept FAD	378
9.3.3	Large Scale Test of a Piping with Pre-cracked Weldments	382
9.3.4	FE-Analysis of Large Scale Piping Test	386
	References	389
	Appendix: Fundamentals of Strength of Materials	391
	Index	443

Glossary

α	Hardening coefficient
α_{cf}	Plastic constraint factor
α_d	Dilatation wave ratio
α_{ij}	Anisotropic elastic constants
α_t	Coefficient of thermal expansion
α^t, α_{ij}^t	Coefficients of thermal expansion tensor
α_s	Shear wave ratio
β_T	Biaxial parameter
β_{ij}	Thermal stress coefficients
β, β_B	Internal hybrid Ansatz coefficients
Γ	Integration path
Γ^+, Γ^-	Upper, lower crack faces
Γ_ε	Crack tip integration path
γ	Sliding
γ	Material constant
γ	Specific surface energy
γ_I	Principal shear strains
γ_{II}	Principal shear strains
γ_{III}	Principal shear strains
γ_d	Aspect ratio dilatation waves
γ_D	Dynamic surface energy
γ_s	Aspect ratio shear waves
γ_t	Crack opening angle
$\Delta\sigma$	Cyclic stress range
ΔK	Cyclic stress intensity factor
ΔK_{eff}	Effective cyclic stress intensity
ΔK_{th}	Threshold value fatigue
δ	Variational symbol
δ	Separation (cohesive zone model)
δ_c	Decohesion length
δ_n	Separation (normal)

δ_s	Separation (transversal)
δ_t	Separation (tangential)
δ_t	Crack opening displacement CTOD
δ_T	Total shear separation
$\boldsymbol{\delta} = [\mathbf{u}]$	Separation vector (cohesive model)
ϵ	Dielectric constant
ϵ	Bimaterial constant
ϵ_{ijk}	Permutation tensor LEVI — CEVITA
ϵ_0	Reference strain ($\approx \sigma_{F/E}$)
ϵ_I	Principal strain
ϵ_{II}	Principal strain
ϵ_{III}	Principal strain
ϵ^H	Dilatational strain tensor
ϵ_v^p	Equivalent plastic strain
ϵ_M^p	Equivalent plastic matrix strain
$\boldsymbol{\epsilon}, \epsilon_{ij}$	Strain tensor
$\boldsymbol{\epsilon}^D, \epsilon_{ij}^D$	Strain deviator
$\boldsymbol{\epsilon}^e, \epsilon_{ij}^e$	Elastic strains
$\boldsymbol{\epsilon}^p, \epsilon_{ij}^p$	Plastic strains
$\boldsymbol{\epsilon}^t, \epsilon_{ij}^t$	Thermal strains
$\boldsymbol{\epsilon}$	Strain matrix
$\boldsymbol{\epsilon}^e$	Elastic strain matrix
$\boldsymbol{\epsilon}^p$	Plastic strain matrix
$\boldsymbol{\epsilon}^*$	Initial strain matrix
ζ	Complex variable
η	Shear-tension ratio (cohesive model)
η	Error indicator FEM global
$\eta(x_1)$	Gradient function
$\eta(a/w)$	Geometry function J_p -integral
η_e	Error indicator element e
$\boldsymbol{\eta}, \eta_{mn}$	EULER-ALMANSI strain tensor
θ	Polar coordinate, angle
θ_c	Crack propagation angle
θ_d	Angle for dilatation waves
θ_s	Angle for shear waves
ϑ	Heat transition coefficient
κ	Elastic constant
χ	Node distortion parameter
χ	Crack tip position
χ	Dynamic exaggeration factor
$\dot{\Lambda}$	Plastic LAGRANGE multiplier
λ	Exponent of complex stress function
λ	LAME'S elasticity constant

μ	Shear modulus
μ	Shear shape factor
ν	POISSON'S number
ξ, ζ_i	Natural element coordinates
ξ^g, ζ_i^g	Coordinates of integration points
Π_C	Principle of complementary energy
Π_{CH}	Hybrid stress principle
Π_{GH}	Hybrid mixed principle
Π_{MH*}	Simplified hybrid mixed principle
Π_P	Principle of potential energy
Π_{PH}	Hybrid displacement principle
Π_R	HELLINGER-REISSNER principle
Π_{ext}	Potential of external loads
Π_{int}	Internal mechanical potential
$\hat{\Pi}_{ext}$	Complementary external potential
$\hat{\Pi}_{int}$	Complementary internal potential
ρ	Notch radius
ρ	Density (current configuration)
ρ_0	Density (reference configuration)
ϱ	Mixed-mode-ratio
σ	Normal stress (cohesive zone model)
σ_c	Cohesive strength tension
σ_0	Reference stress ($\approx \sigma_F$)
σ_F	Yield stress
σ_{F0}	Initial yield stress
σ^H	Dilatational stress tensor
σ_I	Principal normal stress
σ_{II}	Principal normal stress
σ_{III}	Principal normal stress
σ_c	Critical stress
σ_M	Matrix yield stress
σ_{max}	Maximum stress
σ_{min}	Minimum stress
σ_n	Nominal tensile stress
σ_v	V. MISES equivalent stress
$\boldsymbol{\sigma}, \sigma_{ij}$	CAUCHY'S stress tensor
$\boldsymbol{\sigma}^D, \sigma_{ij}^D$	Stress deviator
$\boldsymbol{\sigma}$	CAUCHY-stress matrix
τ	Shear stress
τ_c	Cohesive strength shear
τ_t	Shear stress tangential
τ_s	Shear stress transversal
τ_F	Shear yield stress
τ_{F0}	Initial shear yield stress

τ_I	Principal shear stress
τ_{II}	Principal shear stress
τ_{III}	Principal shear stress
τ_{ij}	Shear stress components
τ_n	Nominal shear stress
Φ	Yield condition, dissipation function
φ	Electric potential
φ	Angular coordinate for elliptical cracks
φ	Scalar wave potential
$\phi(z)$	Complex stress function
$\chi(z)$	Complex stress function
χ	Crack opening function
ψ	Phase angle
ψ_e	Elastic potential
ψ, ψ_i	Vectorial wave potential
$\Omega(z)$	Complex stress function
$\bar{\Omega}$	Integration domain J -integral
$\bar{\Omega}$	Integration domain J -integral
ω	Damage variable
$\bar{\omega}$	Surface charge density
A	Complex stress coefficient
A	Crack face
A	Surface (reference configuration)
A_I	Factors energy release rate
A_{II}	Factors energy release rate
A_{III}	Factors energy release rate
A_σ	Stress coefficient
A_B	Fracture process zone
A_i	Coefficients of eigenfunctions
$\mathbf{A}^{(e)}$	Correlation matrix
a	Surface (current configuration)
a	Crack length
a	Semi-axis of elliptical cracks
\dot{a}	Crack velocity
\ddot{a}	Crack acceleration
a_0	Initial crack length
a_c	Critical crack length
a_{eff}	Effective crack length
a_i	Coefficients of eigenfunctions
a_{th}	Crack length from threshold value
\mathbf{a}, a_i	Acceleration vector
B	Specimen thickness
B	Complex stress coefficient
B_I	BUECKNER-singularity

B	Strain-displacement-matrix
$\bar{\mathbf{B}}$	Non-linear strain-displacement matrix
$\tilde{\mathbf{B}}$	Hybrid element matrix
b	Ligament length
b_i	Coefficient of eigenfunctions
b_T	Biaxial loading parameter
\mathbf{b}, b_{mn}	Left CAUCHY-GREEN deformation-tensor
$\bar{\mathbf{b}}, \bar{b}_i$	Body force vector
C	Closed Integration path
C	Complex stress coefficient
C	PARIS-coefficient
\mathbf{C}, C_{MN}	Right CAUCHY-GREEN deformation tensor
\mathbf{C}	Material matrix
\mathbf{C}^e	Elasticity matrix
\mathbf{C}^{ep}	Elastic-plastic material matrix
$C_{\alpha\beta}$	Elasticity matrix
\mathbb{C}, C_{ijkl}	Elasticity tensor 4th order
c	Larger semi-axis of elliptical cracks
c_d	Dilatational wave velocity
c_i	Coefficients eigenfunctions
c_R	RAYLEIGH's wave velocity
c_s	Shear wave velocity
c_v	Specific heat capacity
\mathcal{D}	Dissipation energy
D	Plate stiffness
$D(\dot{a})$	RAYLEIGH's function
\mathbf{D}, D_i	Electric flux density
\mathbf{D}	Differentiation matrix
d_p	Extension of plastic zone
dA	Area element
dS	Surface element
dV	Volume element
ds	Line element
\mathbf{d}, d_{ij}	Deformation velocity tensor
E	Elasticity modulus
$E(k)$	Elliptical integral 2. kind
\mathbf{E}, E_{MN}	GREEN-LAGRANGE strain tensor
\mathbf{E}, E_i	Electric field strength
\mathbf{E}	GREEN-LAGRANGE strain matrix
\mathbf{e}_i	Basis vectors
e	EULER number $e \approx 2.718$
F	Single force
$F(\mathbf{x})$	AIRY's stress function
F_L	Plastic limit load (collapse load)

$\tilde{F}_i^{(n)}$	Eigenfunctions mode I
\mathbf{F}, F_{mM}	Deformation gradient
\mathbf{F}	System load vector
\mathcal{F}	Flux integral
f	Void volume fraction
f^*	Modified void volume fraction
f_0	Initial void volume fraction
f_c	Critical void volume fraction
f_f	Void volume fraction at failure
f_N	Void density for nucleation
f_{ij}^L	Angular functions crack tip fields ($L = \text{I, II, III}$)
\mathbf{f}	Element load vector
G	Energy release rate
G_I	Energy release rate für crack mode I, II, III
G_{II}	Energy release rate für crack mode I, II, III
G_{III}	Energy release rate für crack mode I, II, III
G_c	Critical energy release rate
G^{dyn}	Dynamic energy release rate
G_i^I, G_i^{II}	Fracture mechanical weight functions
$\tilde{G}_i^{(n)}$	Eigenfunctions mode II
\mathbf{G}	Hybrid element matrix
$g(a, w)$	Geometry function for K -factors
g_i^L	Angular functions crack tip fields ($L = \text{I, II, III}$)
\mathbf{g}, g_i	Temperature gradient
H	Height crack element
$H(\tau)$	HEAVISIDE's jump function
H_α	Hardening function
H_i^I, H_i^{II}	Fracture mechanical weight functions
H_{ij}	IRWIN-matrix anisotropy
$\tilde{H}_3^{(n)}$	Eigenfunctions Modus III
\mathbf{H}	Hybrid element matrix
\mathbf{H}	Matrix hardening function
$\bar{\mathbf{H}}$	Displacement gradient matrix
h	Thickness (plates, sheets)
\tilde{h}	Stress triaxiality number
h_α	Hardening variable
\mathbf{h}, h_i	Heat flux vector
\mathbf{h}	Matrix hardening variables
I_1^A, I_2^A, I_3^A	Invariants of tensors \mathbf{A}
\mathbf{I}, δ_{ij}	KRONECKER's symbol, unity tensor
\mathbf{I}_p, I_{pi}	Momentum vector
$i = \sqrt{-1}$	Imaginary unit number
J	J -integral

J	Determinant of deformation gradient $\det \mathbf{F} $
J^*	Dynamic J -integral (stationary crack)
J^{dyn}	Dynamic J -integral (moving crack)
\hat{J}	3D disk-shaped integral
J_{Ic}	Critical material parameter
$J_{\text{R}}(\Delta a)$	Crack resistance curve (EPFM)
J_{e}	Elastic part of J
J_{p}	Plastic part of J
\mathbf{J}, J_k	J -integral vector
$\tilde{\mathbf{J}}, \tilde{J}_k$	Elastic-plastic J -integral
$\mathbf{J}^{\text{te}}, J_k^{\text{te}}$	Thermoelastic J -integral
\mathbf{J}	JACOBI'S functional matrix
\mathcal{K}	Kinetic energy
K	Compression modulus
K_D	Dielectric displacement intensity factor
K_{I}	Stress intensity factors
K_{II}	Stress intensity factors
K_{III}	Stress intensity factors
K_{I}^{d}	Dynamic stress intensity factor
$K_{\text{Ic}}, K_{\text{IIc}}$	Static fracture toughness
K_{ID}	Dynamic fracture toughness (moving crack)
K_{Id}	Dynamic fracture toughness (stationary crack)
K_{Ia}	Crack arrest toughness
K_{max}	Maximum stress intensity
K_{min}	Minimum stress intensity
K_{op}	Crack opening intensity factor
K_{v}	Equivalent stress intensity factor
K_1, K_2	Stress intensity factors for interface crack
\tilde{K}	Complex stress intensity factor
\mathbf{K}	System stiffness matrix
k	Specimen stiffness
k	Heat conduction coefficient
k_1, k_2	Stress intensity factors for plates
\mathbf{k}	Element stiffness matrix
L	Length crack element
$\mathcal{L}(u, \dot{u}, t)$	LAGRANGE'S function
$\tilde{L}_{ij}^{(n)}$	Eigenfunctions Mode III
\mathbf{L}	Hybrid displacement matrix
\mathbf{l}, l_{ij}	Velocity gradient
dL	Line element length (reference configuration)
dl	Line element length (current configuration)
Δl_k	Virtual displacement of crack front
$\tilde{M}_{ij}^{(n)}$	Eigenfunctions Mode I

M	System mass matrix
m	PARIS-exponent
m_i	Bending moments (plate theory)
m	Element mass matrix
N	Number of load cycles
N_K	Number of all nodes FEM-system
$N_a(\xi_i)$	Shape functions
N_B	Load cycles until fracture
$\tilde{N}_{ij}^{(n)}$	Eigenfunctions Mode II
\hat{N}, \hat{N}_{ij}	Normal direction in stress space
N	Matrix of shape functions
n_D	Number of dimensions
n_E	Number of finite elements
n_G	Number of GAUSS-points
n_H	Number of hardening variables
n_K	Number of nodes per element
n_L	Number of individual forces
n_f	Number of rigid body degrees of freedom
n , n_i	Normal vector
P	Global load parameter
P	Crack face forces
P, P_k	Generalized configurational force
P, P_{Mn}	1st PIOLA-KIRCHHOFF stress tensor
P	Hybrid stress matrix
$p(x_1)$	Crack face loads
$p(\mathbf{x})$	Surface loads (plate theory)
p, p_i	Material body force vector
Q	Thermal energy
Q	Constraint factor (EPFM)
Q	Crack face forces
Q, Q_{ij}	Energy-momentum-tensor
q	Displacement of load point
q	Weighting function 2D
$q(x_1)$	Crack face loads
q_1, q_2, q_3	Parameters GURSON-Model
q_i	Transversal forces (plate theory)
q_k	Weighting function 3D
R	Stress ratio K_{\min}/K_{\max}
$R(\hat{\varepsilon}^p)$	Isotropic hardening variable
R_m	Ultimate tensile strength
$R_{p0,2}$	Yield strength
$R(\Delta a)$	Crack growth resistance curve (LEFM)
R , R_{nM}	Rotation tensor
R	Hybrid boundary stress matrix

R	Residual vector
r	Polar coordinate, radius
r_B	Size of fracture process zone
r_F	Radius plastic Zone
r_J	Dominance radius J-field
r_K	Dominance radius K-field
r_p	Size of plastic zone
r_d	Radius of dilatation waves
r_s	Radius of shear waves
\mathbf{r}, r_{ij}	Transformation matrix for rotation
S	Surface
\tilde{S}	Interelement boundary
$S(\theta)$	Energy density factor
S^+, S^-	Upper, lower crack surface
S_e	Surface crack tube
S_{end}	End faces
S_t	Boundary with given $\bar{\mathbf{t}}$
S_u	Boundary with given $\bar{\mathbf{u}}$
\mathbf{S}, S_k	Sectional force (reference configuration)
$\mathbb{S}, \mathbb{S}_{ijkl}$	Elastic compliance tensor
\mathbf{S}	Elastic compliance matrix
s	Arc length
\mathbf{s}, s_k	Sectional force (current configuration)
T	Temperature field
T_{ij}	Stress components 2nd order
T_k^*	Generalized energy integral
\mathbf{T}, T_{MN}	2nd PIOLA-KIRCHHOFF stress tensor
\mathbf{T}	2nd PIOLA-KIRCHHOFF stress matrix
\mathbf{t}, t_i	Sectional traction vector
$\bar{\mathbf{t}}, \bar{t}_i$	Traction vector
$\bar{\mathbf{t}}$	Boundary stress matrix
\mathbf{t}	Traction vector (cohesive zone model)
t^c, t_i^c	Crack face tractions
U	Strain energy density
U	Crack opening factor
\hat{U}	Complementary strain energy density
\check{U}	Specific stress work density
U^e	Elastic strain energy density
U^p	Plastic stress work density
\check{U}^{te}	Thermoelastic strain energy density
\mathbf{U}, U_{MN}	Right stretch tensor
\mathbf{u}, u_i	Displacement vector
$\tilde{\mathbf{u}}, \tilde{u}_i$	Element boundary displacements
$\bar{\mathbf{u}}, \bar{u}_i$	Boundary displacement vector

\mathbf{u}	Displacement matrix
V	Notch opening displacement COD
V	Volume (reference configuration)
\mathbf{V}, V_{mn}	Left stretch tensor
\mathbf{V}	System nodal displacements
v	Volume (current configuration)
\mathbf{v}, v_i	Velocity vector
\mathbf{v}	Nodal displacements vector
\mathcal{W}_{ext}	External mechanical work
\mathcal{W}_{int}	Internal mechanical work
$\widehat{\mathcal{W}}_{\text{ext}}$	Complementary external work
$\widehat{\mathcal{W}}_{\text{int}}$	Complementary internal work
\mathcal{W}_c	Work for crack opening
\mathcal{W}_B	Work in fracture process zone
w	Specimen width
$w(\mathbf{x})$	Deflection (plate theory)
w_g	Weighted integration rules
\mathbf{w}, w_{ij}	Spin tensor
\mathbf{X}, X_M	Coordinates (material)
\mathbf{X}, X_{ij}	Kinematic hardening variable
\mathbf{x}, x_m	Coordinates (spatial)
\mathbf{x}	Element coordinate matrix
$\hat{\mathbf{x}}$	Nodal coordinate matrix
z	Complex variable
z	Arbitrary FEM output quantity
1D	One-dimensional
2D	Two-dimensional
3D	Three-dimensional
ASTM	American Society for Testing of Materials
BVP	Boundary Value Problem
CT	Compact Tension Specimen
CTE	Crack Tip Element
CTOA	Crack Tip Opening Angle
CTOD	Crack Tip Opening Displacement
DIM	Displacement Interpretation Method
EDI	Equivalent Domain Integral
EPFM	Elastic-Plastic fracture mechanics
ESIS	European Structural Integrity Society
FAD	Failure Assessment Diagram
FEM	Finite Element Method
IBVP	Initial Boundary Value Problem
LEFM	Linear-Elastic Fracture Mechanics
LSY	Large-Scale Yielding
MCCI	Modified Crack Closure Integral

ODE	Ordinary Differential Equation
PC	Plastic Collapse
PDE	Partial Differential Equation
QPE	Quarter-Point Elements
RSE	Regular Standard Elements
SENB	Single Edge Notched Bending Specimen
SINTAP	Structural Integrity Assessment procedure
SSY	Small-Scale Yielding
SZH	Stretched Zone Height
VCE	Virtual Crack Extension