Continuum Damage Mechanics

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Volume 185

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Sumio Murakami

Continuum Damage Mechanics

A Continuum Mechanics Approach to the Analysis of Damage and Fracture

Springer

Sumio Murakami
Nagoya University
Professor Emeritus
3-63-3 Hinata-cho, Mizuho-ku
467-0047 Nagoya
Japan
sy-murakami@wb3.so-net.ne.jp

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Preface

A half century has passed since Professor L. M. Kachanov, a Russian authority of nonlinear solid mechanics, proposed a quantitative notion of *Damage* in order to predict the brittle creep rupture time of metals for the first time. Then, together with Professor Y. N. Rabotnov, another authority in Russia in this field, he laid the basis of a new branch of *Damage Mechanics* or *Continuum Damage Mechanics*.

Starting with metallic materials, the objective of damage mechanics research has been extended thereafter to concrete, geological materials, polymers, composites and other materials. The extent of problems discussed by damage mechanics has been enlarged to a wide variety of damage phenomena including elastic-plastic damage, elastic-brittle damage, fatigue damage, dynamic and spall damage, etc.

Damage mechanics which originally started as a phenomenological theory of damage and fracture has been reinforced logically by the help of well established theoretical frameworks of material science, nonlinear continuum mechanics, irreversible thermodynamics, micromechanics, computational mechanics, etc., and now has been established as a precise and systematic discipline for damage and fracture analysis. The results of the development of this field of mechanics have been published in several excellent books and treatises. However, some of them may be readable only for readers with specially trained scientific background.

Recent developments in engineering and technology have brought about serious and enlarged demands for the reliability, safety and economy in wide field of science, ranging from aeronautics, civil and structural engineering to automotive and production engineering. This, in turn, has caused more interest in continuum damage mechanics and its engineering applications.

This book aims to give a concise overview of the current state of damage mechanics, and then to show the fascinating possibility of this promising branch of mechanics. The author, therefore, intended to provide researchers, engineers and graduate students of various mechanical grounding with an intelligible and self-contained textbook to study this rational and fruitful mechanics.

The book consists of two Parts and an Appendix. Part I is concerned with the foundation of continuum damage mechanics. Basic concept of material damage and the mechanical representation of damage state of various kinds are described in Chapters 1 and 2. In Chapters 3 through 5, irreversible thermodynamics, thermodynamic constitutive theory and its application to the modeling of the constitutive and the evolution equations of damaged materials are described as a systematic basis for the subsequent development throughout the book.

Part II describes the application of the fundamental theories developed in Part I to typical damage and fracture problems encountered in various fields of current engineering. Important engineering aspects of elastic-plastic or ductile damage, their damage mechanics modeling and their further refinement are first discussed in Chapter 6. Chapters 7 and 8 are concerned with the modeling of fatigue, creep, creep-fatigue and their engineering application. Damage mechanics modeling of complicated crack closure behavior in elastic-brittle and composite materials are discussed in Chapters 9 and 10. In Chapter 11, applicability of the local approach to fracture by means of damage mechanics and finite element method, and the ensuing mathematical and numerical problems are briefly discussed.

A proper understanding of the subject matter requires the knowledge of tensor algebra and tensor calculus. At the end of this book, therefore, foundations of tensor analysis are presented in the Appendix, especially for readers with insufficient mathematical background, but with keen interest in this exciting field of mechanics.

For the publication of this book, the author has been greatly indebted to many people for their enlightening, guidance and support. The author is grateful particularly to Professor J. Lemaitre at LMT (Laboratoire de Mécanique et Technologie), ENS (Ecole Normal Supérieure) de Cachan, France, for his lifelong friendship and stimulation to the academic activity of the author. In 1970s, Professor Lemaitre started his damage mechanics research together with Dr. J. L. Chaboche, currently at ONERA (Office National d'Études et de Recherches Aérospatiales), and other excellent young researchers, and established the current foundation of damage mechanics. The basic theories and their application described in this book are largely due to their research results. Dr. Chaboche, in particular, has pursued the extension of the applicability of continuum damage mechanics, and derived a number of rigorous theoretical frameworks for this mechanics.

Professor D. R. Hayhurst, the University of Manchester, UK, also made a fundamental contribution, especially to the development of creep damage theory, and damage analysis of high temperature structures. His work in the late 1960s laid the foundations for the development of computational Continuum Damage Mechanics; and additionally, in the 1970s, he established with Professor F. A. Leckie important bounding theorems for structures. The author owes a considerable part of this book, especially those in Chapter 8, to Professor Hayhurst's results.

Dr. Chaboche and Professor Hayhurst, in their very busy days of scientific activity, spent a lot of time in reading through the draft of this book, and gave a number of valuable suggestions for its improvements. The author is deeply grateful to them for their favor.

The author greatly acknowledges the contribution of Professor C. L. Chow, University of Michigan, Dearborn, USA. Working with able researchers, mainly from Asia, he established an accurate and firm basis for anisotropic elastic-plastic damage theory. Professor Chow's effort to start and to develop the *International Journal of Damage Mechanics* should be deeply appreciated. A number of his research results are referenced in this book.

The author pays his heartfelt respect and gratitude to the late Professor D. Krajcinovic, Arizona State University, USA, for his friendship and his contribution to damage mechanics. He made a magnificent contribution to the development of this discipline for geological materials, mainly from a micromechanics point of view. He left an everlasting excellent treatise of this subject.

In the course of writing and improving the manuscript, the author is indebted largely to a number of foreign colleagues and friends, especially to Professor P. Ladevèze, ENS/LMT, Professor K. Saanouni, Université de Tchnologie de Troyes and Professor J. Mazars, Institut National Polytechnique de Grenoble. Besides references to their outstanding research results, their invaluable comments and suggestions to improve the draft of this book are greatly appreciated.

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Finally special thanks should be expressed to Springer-Verlag for their favor to undertake the publication of this book. Sincere gratitude is due to Ms. N. Jacobs, Publishing Editor, and Ms. J. Pot, Editor at Springer-Dordrecht for their cooperation and assistance in printing the book in this excellent form. The author gratefully acknowledges the favor and endeavor of Mr. K. Kimlica, Springer-Tokyo, who gave the author the unique chance to publish this book with the well established and prestigious publishing house of Springer.

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Sumio Murakami

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List of Symbols

Symbols for Tensors

Light-face Letters	$a, b, \ldots, A, B, \ldots, \mathscr{A}, \mathscr{B}, \ldots,$ $\alpha, \beta \ldots, \Gamma, \Delta, \ldots$: scalars
Lowercase bold-face Latin letters	$\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \ldots$: vectors
Uppercase bold-face Latin letters	$\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \ldots$: second-order tensors
Uppercase blackboard Latin letters	$\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots$: fourth-order tensors
Uppercase calligraphic Latin letters	$\boldsymbol{\mathcal{A}}, \boldsymbol{\mathcal{B}}, \boldsymbol{\mathcal{C}}, \ldots$: tensors of arbitrary-order

Symbols of Latin Letters

A

$\mathbb{A}^F, \mathbb{A}^M$: tensors of strain-concentration factors in fiber and matrix of a composite
$\boldsymbol{A}$: kinematic hardening variable (or back stress tensor)
$\boldsymbol{A}_k$: generalized force vector
dA: area of surface element
$d\tilde{A}$: effective area of surface element
dA_D: reduction in surface element due to voids
$\boldsymbol{a}_p$: basis vectors of six-dimensional space

$$\{\boldsymbol{a}_p\} = \left\{\boldsymbol{e}_{ij}^{\mathrm{S}}\right\} = \left\{\left(\boldsymbol{e}_i \otimes \boldsymbol{e}_j\right)^{\mathrm{S}}\right\}$$

a: crack length

B

$\mathscr{B}^F$, $\mathscr{B}^M$: stress-concentration factors in fiber and matrix of a composite
$\mathbb{B}^F, \mathbb{B}^M$: tensors of stress-concentration factors in fiber and matrix of a composite
B: body, region occupied by a body, configuration of a body, damage-strengthening variable associated with β

B_f, B_t: fictitious undamaged and current damaged configurations of a body
∂B: boundary, surface of body B

C

$\mathbb{C}_0, \mathbb{C}(\boldsymbol{\mathcal{D}})$: elastic modulus tensors of undamaged and damaged materials
$\mathbb{C}_{AC}$: activated elastic modulus tensor
$\mathbb{C}_{EF}$: effective elastic modulus tensor
C_{pq}: components of elastic modulus matrix in Voigt notation
C_{pq}: components of elastic modulus matrix in matrix representation of an elastic constitutive equation $\left[\mathrm{C}_{pq}\right] = \left[C_{pr}\right]\left[W_{rq}\right]$
$c^F,\ c^M$: volume fractions of fiber and matrix in a composite

D

$\boldsymbol{\mathcal{D}}, \mathbb{D}, \boldsymbol{D}, D$: damage tensors of an arbitrary-order, fourth-, second- and 0th-order
D_C: critical value of $\boldsymbol{\mathcal{D}}$ for fracture initiation
$D_1,\ D_2,\ D_F,\ D_S,\ D_T$: scalar damage variables
$D_0,\ D_{ij},\ D_{ijk\ell},\ \ldots$: fabric tensors

E

$\mathscr{E}^n$: n-dimensional Euclidean vector space
E: internal energy of a system
$E,\ E_0,\ E_i,\ E_{ij}$: Young's modulus (or modulus of elasticity)
$\boldsymbol{e}_i$: orthonormal basis vectors
$\boldsymbol{e}_{ij}$: second-order basis tensors $\boldsymbol{e}_{ij} = \boldsymbol{e}_i \otimes \boldsymbol{e}_j$
e: internal energy per unit mass
e_{ijk}: permutation tensor (or Eddington epsilon, or Levi-Civita symbol)

F

F: dissipation potential, force, load, yield function
$\boldsymbol{f}$: body force vector
f: yield function, void volume fraction
f_C: critical void volume fraction

G

G: damage variable
$G,\ G_i,\ G_{ij}$: shear modulus
$\boldsymbol{g}$: temperature gradient $\boldsymbol{g} = -\mathrm{grad}T$

H

H: strain hardening rate, damage variable
$H(\ )$: Heaviside function $H(x) = 1$ for $x \geq 0$,
$H(x) = 0$ for $\boldsymbol{x} < 0$
h: enthalpy per unit mass
$h(\boldsymbol{s} - \boldsymbol{x})$: nonlocal weighting function

I

$\mathbb{I}, \bar{\mathbb{I}}$: forth-order identity tensors
$\mathbb{I}^{\mathrm{S}}$: fourth-order identity tensor transforming a second-order symmetric tensor to itself
$\boldsymbol{I}$: identity (or unit) linear transformation, second-order identity (or unit) tensor
I_1, I_2, I_3: principal invariants of a second-order tensor

J

$\boldsymbol{J}$: generalized flux vector

K

K: bulk modulus, kinetic energy of a system

L

L: segment
L_{KM}: coefficients in the phenomenological relation of Onsager
ℓ: characteristic length

M

$\mathbb{M}$: damage effect tensor
M: microscopic volume element

N

$\boldsymbol{N}_p$: basis of six-dimensional vector space spanned by the dyad $\boldsymbol{n}_i \otimes \boldsymbol{n}_j$ of principal damage directions
N: number of loading cycles
N_C: number of cycles to fracture due to creep damage
N_D: number of cycles to microcrack initiation

N_F: crack extension life (number of cycles for a microcrack to grow to a crack of mesoscale)
N_R: fatigue life (number of cycles to fatigue fracture)
$\boldsymbol{n}$: unit normal vector
$\boldsymbol{n}_i$: vectors of principal directions of a second-order tensor
n: creep exponent, strain hardening exponent

O

$\boldsymbol{O}$: zero linear transformation, zero tensor
O: origin of a coordinate system

P

$\mathcal{P}^3$: three-dimensional Euclidean point space
$\mathbb{P}_D$: fourth-order projection tensor of a principal damage direction $\boldsymbol{n}^D$

$$\mathbb{P}_D = \boldsymbol{n}^D \otimes \boldsymbol{n}^D \otimes \boldsymbol{n}^D \otimes \boldsymbol{n}^D$$

$\mathbb{P}_i$: fourth-order projection tensor of a direction $\boldsymbol{n}_i$

$$\mathbb{P}_i = \boldsymbol{n}_i \otimes \boldsymbol{n}_i \otimes \boldsymbol{n}_i \otimes \boldsymbol{n}_i$$

$\mathbb{P}_\varepsilon^+$: positive orthogonal projection tensor of strain tenor $\boldsymbol{\varepsilon}$

$$\mathbb{P}_{\boldsymbol{\varepsilon}}^+ \equiv \sum_{i=1}^{3} \sum_{j=1}^{3} H(\varepsilon_i)\, H\left(\varepsilon_j\right) \delta_{ik} \delta_{jl} \boldsymbol{n}_i^{(\varepsilon)} \otimes \boldsymbol{n}_j^{(\varepsilon)} \otimes \boldsymbol{n}_k^{(\varepsilon)} \otimes \boldsymbol{n}_l^{(\varepsilon)}$$

$\mathbb{P}_\varepsilon^-$: negative orthogonal projection tensor of strain tensor $\boldsymbol{\varepsilon}$

$$\mathbb{P}_\varepsilon^- = \mathbb{I} - \mathbb{P}_\varepsilon^+$$

$\mathbb{P}_\sigma^+$: positive orthogonal projection tensor of stress tensor $\boldsymbol{\sigma}$

$$\mathbb{P}_{\boldsymbol{\sigma}}^+ \equiv \sum_{i=1}^{3} \sum_{j=1}^{3} H(\sigma_i)\, H\left(\sigma_j\right) \delta_{ik} \delta_{jl} \boldsymbol{n}_i^{(\sigma)} \otimes \boldsymbol{n}_j^{(\sigma)} \otimes \boldsymbol{n}_k^{(\sigma)} \otimes \boldsymbol{n}_l^{(\sigma)}$$

$\mathbb{P}_\sigma^-$: negative orthogonal projection tensor of stress tensor $\boldsymbol{\sigma}$

$$\mathbb{P}_\sigma^- = \mathbb{I} - \mathbb{P}_\sigma^+$$

P: a material point, a point in Euclidean point space
p: accumulated equivalent plastic strain $p = \int \dot{\varepsilon}_{EQ}^p dt$

p_D: threshold of accumulated equivalent plastic strain for damage initiation
p_R: accumulated equivalent plastic strain at fracture, equivalent plastic strain at fracture

Q

$\boldsymbol{Q}$: orthogonal tensor, orthogonal transformation
Q: a material point, a point in Euclidean point space
$\dot{Q}$: rate of heat supply to a system
$\boldsymbol{q}$: heat flux vector

R

R: a material point, a point in Euclidean point space
R: isotropic hardening variable, average radius of spherical voids
R_ν: stress triaxiality function

$$R_\nu = (2/3)\,(1+\nu) + 3\,(1-2\nu)\left(\sigma_H/\sigma_{EQ}\right)^{1/2}$$

r: isotropic hardening internal variable, heat generation rate per unit volume

S

$\mathbb{S}_0, \mathbb{S}(\boldsymbol{\mathcal{D}})$: elastic compliance tenors of undamaged and damaged materials
S: boundary, surface, entropy of a system
S_{pq}: components of elastic compliance matrix in Voigt notation
S_{pq}: components of elastic compliance matrix in matrix representation of elastic constitutive equation $\left[\mathrm{S}_{pq}\right] = \left[S_{pr}\right]\left[W_{rq}\right]$
dS: surface element
s: entropy per unit mass

T

$\boldsymbol{T}^\nu$: surface force vector (or traction vector)
T: absolute temperature
$\boldsymbol{t}$: surface force vector (or traction vector)
t: time
t_R: creep rupture time
t_0: time of creep damage initiation

U

$\boldsymbol{u}$: displacement vector
$\boldsymbol{u}^{(1)}, \boldsymbol{u}^{(2)}, \boldsymbol{u}^{(3)}$: eigen vectors (or principal directions) of a second-order tensor

V

$\boldsymbol{V}_k$: vector of internal variables
$\dot{\boldsymbol{V}}_k$: generalized flux vector
$\boldsymbol{V}_p$: internal variable of plastic deformation
V: region occupied by a body, volume of a material
$V_0(\boldsymbol{\sigma}, \boldsymbol{\alpha})$, $V(\boldsymbol{\sigma}, \boldsymbol{\mathcal{D}}, \boldsymbol{\alpha})$: complementary strain energy functions of undamaged and damaged materials
dV, dV_0, dV_ξ: volume elements
$\boldsymbol{v}$: velocity vector

W

W: work done by external forces
W^E: elastic strain energy per unit volume
$W_0(\boldsymbol{\varepsilon}, \boldsymbol{\alpha})$, $W(\boldsymbol{\varepsilon}, \boldsymbol{\mathcal{D}}, \boldsymbol{\alpha})$: strain energy functions of undamaged and damaged materials
W_D: energy stored up to damage initiation
W_S: energy stored in material
W_{pq}: components of weighting matrix
$\dot{W}$: rate of external mechanical work (or external mechanical power)
ΔW: energy per loading cycle

X

$\boldsymbol{X}$: generalized force vector
$\boldsymbol{x}$: position, position vector of a material point
x_i: coordinates

Y

$\mathbb{Y}, \boldsymbol{Y}, Y$: generalized forces associated with $\mathbb{D}$, $\boldsymbol{D}$ and D (or damage-associated variables)

Symbols of Greek Letters

A

$\boldsymbol{\alpha}$: kinematic hardening internal variable (or associated variable of $\boldsymbol{A}$)

B

β: damage-strengthening internal variable (or associated variable of B)

Γ

Γ: Gibbs potential per unit mass
γ_p: components of column vector of engineering strain
γ_{ij}: shear strain

Δ

δ_{ij}: Kronecker delta $\delta_{ij} = 1$ for $i = j$,
$\delta_{ij} = 0$ for $i \neq j$

E

$\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^e, \boldsymbol{\varepsilon}^p, \boldsymbol{\varepsilon}^v$: strain tensors
$\boldsymbol{\varepsilon}^D$: deviatoric strain tensor $\boldsymbol{\varepsilon}^D = \boldsymbol{\varepsilon} - (1/3)\,(tr\boldsymbol{\varepsilon})\,\boldsymbol{I}$
$\boldsymbol{\varepsilon}^+$: positive-valued strain tensor $\boldsymbol{\varepsilon}^+ = \mathbb{P}_\varepsilon^+ : \boldsymbol{\varepsilon}$
$\boldsymbol{\varepsilon}^-$: negative-valued strain tensor $\boldsymbol{\varepsilon}^- = \mathbb{P}_\varepsilon^- : \boldsymbol{\varepsilon}$
ε_{EQ}: equivalent strain $\varepsilon_{EQ} = \left[(2/3)\,\varepsilon_{ij}^D \varepsilon_{ij}^D\right]^{1/2}$
ε_H: mean strain $\varepsilon_H = (1/3)\,\varepsilon_{kk}$
ε_p: strain components in Voigt notation
$\Delta\varepsilon$: strain range

Z

ς: material constant

H

η: material constant characterizing the opening-closing effect of a crack

Θ

θ: angle

Λ

$\dot{\Lambda}$: indeterminate multiplier of plastic and/or damage evolution equation
λ: stress singularity exponent at crack tip

λ: Lamé constant $\lambda = 2G\nu/(1-2\nu)$
$\lambda_1,\ \lambda_2,\ \lambda_3$: eigen values (or principal values) of a second-order tensor

M

μ: Lamé constant $\mu = G$

N

$\boldsymbol{\nu}$: unit normal vector
$\nu,\ \nu_i,\ \nu_{ij}$: Poisson's ratio

Ξ

ξ: void density, void volume fraction
$\xi(\boldsymbol{n})$: distribution function of void density

P

ρ: mass density

Σ

$\boldsymbol{\sigma}$: stress tensor
$\boldsymbol{\sigma}^D$: deviatoric stress tensor $\boldsymbol{\sigma}^D = \boldsymbol{\sigma} - (1/3)(\mathrm{tr}\boldsymbol{\sigma})\boldsymbol{I}$
$\boldsymbol{\sigma}^+$: positive-valued stress tensor $\boldsymbol{\sigma}^+ = \mathbb{P}_\sigma^+ : \boldsymbol{\sigma}$
$\boldsymbol{\sigma}^-$: negative-valued stress tensor $\boldsymbol{\sigma}^- = \mathbb{P}_\sigma^- : \boldsymbol{\sigma}$
σ_{EQ}: equivalent stress $\sigma_{EQ} = \left[(3/2)\,\sigma_{ij}^D\sigma_{ij}^D\right]^{1/2}$
σ_F: fatigue limit
σ_H: mean stress (or hydrostatic stress) $\sigma_H = (1/3)\,\sigma_{kk}$
σ_R: fracture stress
σ_U: tensile strength
σ_V: viscous stress
σ_Y: yield stress
σ_a: stress amplitude
σ_p: stress components in Voigt notation
$\Delta\sigma$: stress range

T

τ: shear stress

Φ

Φ: dissipation per unit volume
$\Phi(r,\ \theta)$: stress function of asymptotic stress field

X

$\chi(\sigma_{ij})$: stress criterion of damage or fracture

Ψ

ψ: Helmholtz free-energy per unit mass
ψ: damage variable $\psi = 1 - D$

Ω

Ω: solid angle, region of three-dimensional Euclidean point space
$\Omega_C,\ \Omega_F$: damage variables of creep and fatigue

Symbols for Operation

$(\cdot)$: scalar product, contraction[1]* $\boldsymbol{u}\cdot\boldsymbol{v} = u_i v_i$
$(:)$: double contraction $\boldsymbol{A}:\boldsymbol{B} = A_{ij}B_{ij}$
$(\times)$: vector product $\boldsymbol{u}\times\boldsymbol{v} = e_{ijk}u_j v_k \boldsymbol{e}_i$
$\otimes$: tensor product
tensor product of vectors $\boldsymbol{a}\otimes\boldsymbol{b} = a_i b_j \boldsymbol{e}_i\otimes\boldsymbol{e}_j$
tensor product of second-order tensors 1

$$\boldsymbol{A}\otimes\boldsymbol{B} = A_{ij}B_{kl}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}_l$$

$\underline{\otimes}$: tensor product of second-order tensor 2

$$\boldsymbol{A}\underline{\otimes}\boldsymbol{B} = A_{ik}B_{jl}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}_l$$

$\overline{\otimes}$: tensor product of second-order tensor 3

$$\boldsymbol{A}\overline{\otimes}\boldsymbol{B} = A_{il}B_{jk}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}_l$$

$\overline{\underline{\otimes}}$: tensor product of second-order tensor 4

$$\boldsymbol{A}\overline{\underline{\otimes}}\boldsymbol{B} = \frac{1}{2}(A_{ik}B_{jl} + A_{il}B_{jk})\boldsymbol{e}_i\otimes\boldsymbol{e}_j\otimes\boldsymbol{e}_k\otimes\boldsymbol{e}_l$$

[1] In the case of scalar product between a tensor and a vector, or between two tensors, the dot (·) representing scalar product is usually omitted by convention. See Eqs. (12.8) through (12.12) in Appendix and the related foot note

∂_i: partial derivative with respect to coordinate x_i $\quad \partial_i = \partial/\partial x_i$
∇: vector operator (or Nabla, or del operator) $\quad \nabla = \partial_i \boldsymbol{e}_i = (\partial/\partial x_i)\, \boldsymbol{e}_i$
grad$f$: gradient of scalar field f $\quad \mathrm{grad}f = \nabla f = (\partial f/\partial x_i)\, \boldsymbol{e}_i$
grad$\boldsymbol{f}$: gradient of vector $\boldsymbol{f}$ $\quad \mathrm{grad}\boldsymbol{f} = \begin{cases} \boldsymbol{f} \otimes \nabla = \left(\partial f_i/\partial x_j\right) \boldsymbol{e}_i \otimes \boldsymbol{e}_j \\ \nabla \otimes \boldsymbol{f} = \left(\partial f_j/\partial x_i\right) \boldsymbol{e}_i \otimes \boldsymbol{e}_j \end{cases}$
div$\boldsymbol{f}$: divergence of vector $\boldsymbol{f}$ $\quad \mathrm{div}\boldsymbol{f} = \boldsymbol{f} \cdot \nabla = \nabla \cdot \boldsymbol{f} = \partial f_i/\partial x_i$
curl$\boldsymbol{f}$: curl (or rotation) of vector $\boldsymbol{f}$

$$\mathrm{curl}\boldsymbol{f} = \begin{cases} \boldsymbol{f} \times \nabla = e_{ijk}\left(\partial f_i/\partial x_j\right) \boldsymbol{e}_k \\ \nabla \times \boldsymbol{f} = e_{ijk}\left(\partial f_j/\partial x_i\right) \boldsymbol{e}_k \end{cases}$$

$(\dot{\ })$: time derivative $\quad \dot{\boldsymbol{a}} = \dfrac{\partial \boldsymbol{a}}{\partial \boldsymbol{t}}$
$(\dot{\ })$: material time derivative $\quad \dot{\mathcal{A}} = \dfrac{D\mathcal{A}}{Dt} = \dfrac{\partial \mathcal{A}}{\partial \boldsymbol{t}} + \dfrac{\partial \mathcal{A}}{\partial x_k} v_k$
det$\boldsymbol{A}$: determinant of tensor $\boldsymbol{A}$ $\quad \det\boldsymbol{A} = \det[\boldsymbol{A}] = \det\left[A_{ij}\right]$
tr$\boldsymbol{A}$: trace of tensor $\boldsymbol{A}$ $\quad \mathrm{tr}\boldsymbol{A} = A_{ii}$
$\boldsymbol{A}^{\mathrm{T}}$: transpose of tensor $\boldsymbol{A}$ $\quad \boldsymbol{Au} \cdot \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{A}^{\mathrm{T}}\boldsymbol{v}$
for arbitrary vectors $\boldsymbol{u}$ and $\boldsymbol{v}$
$|\ \ |$: absolute value (or norm, or magnitude)
[ ]: matrix
$[\![\ ]\!]$: discontinuity
$<\ >$: Macauley bracket $\quad < x >= H(x)\, x$

Symbols for Indices

$(\ )^{\mathrm{A}}$: antisymmetric (or skew-symmetric) part of a tensor
$(\ )^{C}$, $(\ )^{c}$: creep
$(\ )^{D}$: deviatoric tensor, damage
$(\ )^{E}$, $(\ )^{e}$: elasticity
$(\ )^{F}$: fiber of composites
$(\ )^{I}$: isotropic hardening
$(\ )^{IN}$: inelasticity
$(\ )^{K}$: kinematic hardening
$(\ )^{M}$: miscroscopic element, matrix of composites
$(\ )^{P}$, $(\ )^{p}$: plasticity
$(\ )^{\mathrm{S}}$: symmetric part of a tensor
$(\ )^{\mathrm{T}}$: transpose of a tensor
$(\ )^{VP}$: viscoplasticity
$(\ )^{-1}$: inverse tensor, inverse transformation
$(\ )_{C}$: creep
$(\ )_{D}$: threshold for damage initiation
$(\ )_{EQ}$: equivalent stress, equivalent strain
$(\ )_{F}$: fatigue, fiber breakage

$(\ )_H$: mean stress, mean strain
$(\ )_R$: fracture, rupture
$(\ )_S$: shear, stored energy
$(\ )_T$: tension
$(\ )_U$: uniaxial state
$(\tilde{\ })$: effective stress, effective strain, fictitious undamaged configuration
$(\bar{\ })$: similar tensor $\bar{\boldsymbol{S}} = \boldsymbol{Q}\boldsymbol{S}\boldsymbol{Q}^{\mathrm{T}}$, similar vector $\bar{\boldsymbol{u}} = \boldsymbol{Q}\boldsymbol{u}$