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Volume 36

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# TRENDS IN LOGIC

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## VOLUME 36

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# Truth and Falsehood

An Inquiry into Generalized Logical Values

 Springer

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*To our beloved spouses Natalia and Petra*

# Preface

This study is the result of a research enterprise that started in 2003 when Y.S. visited H.W. at Dresden University of Technology as a Friedrich Wilhelm Bessel awardee of the Alexander von Humboldt Foundation. Actually, Robert Meyer's remark that taking semantical values such as *both true and false and neither true nor false* seriously leads to madness is what triggered our interest in moving from Nuel Belnap and Michael Dunn's set of four truth values represented as the powerset of the set of classical truth values to its sixteen-element powerset **16**. We soon realized that Bob's provocative verdict, inspiring as it proved to be, should not inhibit a generalization of Belnap and Dunn's approach. The outcome of our initial co-operation was published as [232, 233], and the central and fundamental structure of our investigations into generalized truth values became the trilattice  $SIXTEEN_3$  defined on **16**.

Since we had generated quite a few problems and ideas for additional research, we decided to apply for a research grant in order to further investigate *The Logic of Generalized Truth Values*. Eventually, the project was funded by the German Research Council (DFG) under grant WA 936/6-1. We are very grateful to the DFG for its generous support, including a sabbatical for H.W. in the summer term 2007. Moreover, we are grateful to Ed Zalta for giving us the opportunity to write a survey on the notion of truth values in the Gestalt of an entry for the *Stanford Encyclopedia of Philosophy*, [238]. This helped us a lot in organizing our project.

As the project continued, it benefited further from a Humboldt connection. In 2008, Sergei Odintsov resumed his Humboldt research fellowship at TU Dresden in order to visit H.W. and to work on, among other things, the axiomatization of truth and falsity entailment in the trilattice  $SIXTEEN_3$ . From general algebraic results, we know that truth and falsity entailment in  $SIXTEEN_3$  in the language with two versions of conjunction, disjunction, and negation (one for truth and the other for falsity) can be finitely axiomatized. The non-constructive existence proof, however, does not provide many clues for finding such an axiomatization. Sergei came up with axiom systems for truth and falsity entailment through extending the language by at least one implication connective. The matrix presentation of algebraic operations in  $SIXTEEN_3$  employed in the development of these axiom

systems gave rise to further investigations, in particular to the development of sequent calculi for truth and falsity entailment and the definition of intuitionistic variants of trilattice logics. Sergei's results are presented and discussed in [Chap. 5](#). Also in 2008, Norihiro Kamide joined the Humboldt family by taking up a Humboldt research fellowship at TU Dresden. Norihiro contributed not only his expertise on sequent calculi resulting in the papers [141, 274] with H.W. but also investigated alternative semantics for trilattice logic. We are grateful to Norihiro for his willingness to collaborate and share his work with us in [Chaps. 6](#) and [7](#).

The logics induced by the trilattice  $SIXTEEN_3$  and its higher-valued extensions obtained by iterated powerset-formation are natural extensions of Nuel Belnap and Michael Dunn's logic of first-degree entailment, often referred to as Belnap's or Belnap and Dunn's useful four-valued logic. We therefore refer to these trilattice structures as *Belnap trilattices*. A trilattice of constructive truth values, isomorphic to  $SIXTEEN_3$ , was first presented in joint work by Y.S. Michael Dunn, and Tatsutoshi Takenaka [231]. In fact, the very idea of a truth value trilattice was developed in 1999–2000 during a Fulbright research stay of Y.S. with Michael Dunn at Indiana University. We are most grateful to Nuel Belnap and Michael Dunn for their insights and inspiration, which have been fundamental for our project. Moreover, H.W. is grateful to Nuel for joining him in a reply to critical comments on generalized truth values recently put forward by Didier Dubois, see [273].

In the course of the development of the present inquiry, several people (in addition to ones already mentioned) assisted us in various ways. First of all, we would like to thank Alexander Deck, who not only promptly provided us with all the literature we needed, but also programmed a validity tester for truth and falsity entailment in  $SIXTEEN_3$  and prepared the website for our *2008 International Workshop on Truth Values*. Moreover, Christa Schröder, Konstantin Kleinichen, Andrea Kruse and Caroline Semmling helped us with the workshop. We would like to thank all the participants of this workshop for presenting their work and sharing ideas with us, the extremely sunny weather notwithstanding. In addition to the Humboldt Foundation, we would like to thank the German Society for Analytic Philosophy (GAP) and the Gesellschaft von Freunden und Förderern der Technischen Universität Dresden for their sponsorship of the workshop and Jacek Malinowski, the editor-in-chief of *Studia Logica*, for supporting the publication of two special issues of *Studia Logica* that evolved from the workshop, see [236, 237]. We are also grateful to the Alexander von Humboldt Foundation and the Gesellschaft von Freunden und Förderern der Technischen Universität Dresden for supporting Y.S.'s visit to Dresden University in 2010.

Furthermore, we would like to thank the audiences of various conferences and talks where we had an opportunity to present the results of our research project, including, in the case of H.W., *Logik und Wissen*, Darmstadt, June 2005; *The 49th Annual Meeting of the Australian Mathematical Society*, Perth, September 2005; *Trends in Logic IV. Towards Mathematical Philosophy*, Toruń, September 2006; *GAP.6, Philosophie—Grundlagen und Anwendungen*, Berlin, September 2006; *LOGICA 2007*, Hejnice, June 2007; *ASL Logic Colloquium 2007*, Wrocław,

July 2007; *CLE 30 YEARS/XV Brazilian Logic Conference/XIV Latin-American Symposium on Mathematical Logic*, Paraty, May 2008; *ICCL Summer School 2008*, Dresden, September 2008; *Lebenswelt und Wissenschaft. XXI. Deutscher Kongress für Philosophie*, Essen, September 2008; *Logics of Consequence: A Celebration of Nuel Belnap's Work in Philosophical Logic*, Pittsburgh, April 2009; *8th Smirnov Readings in Logic*, Moscow, June 2009; *LOGICA 2009*, Hejnice, June 2009; *GAP.7, Nachdenken und Vordenken—Herausforderungen an die Philosophie*, Bremen, September 2009; *Applications of Logic in Philosophy and Foundations of Mathematics 2010*, Szklarska Poręba, May 2010; talks at the Australian National University Canberra, September 2006; the University of Melbourne, September 2006; the Institute of Philosophy of the National Academy of Sciences, Kiev, May 2009, and at the University of Göttingen, May 2009. Y.S. presented lectures inter alia at the Institute of Logic and Philosophy of Science, Leipzig University, July 2004; the Institute of Philosophy, Humboldt University Berlin, November 2005; *7th Smirnov Readings in Logic*, Moscow, June 2007; *8th Smirnov Readings in Logic*, Moscow, June 2009; and *GAP.7, Nachdenken und Vordenken—Herausforderungen an die Philosophie*, Bremen, September 2009.

In the present book we, in a sense, summarize and systematize the results of our joint research project, many of which have been published during the last few years. More concretely, [Chap. 1](#) contains material from our entry on Truth Values in the *Stanford Encyclopedia of Philosophy* [238]. [Chapter 2](#) makes use of our analysis of the slingshot argument presented in [235]. In [Chaps. 3](#) and [4](#) we recapitulate the motivating ideas, basic definitions, and results presented in [232] and [233]. As already mentioned, we have also greatly benefited from co-operation with Norihiro Kamide. [Chapter 6](#) on sequent systems for trilattice logics is based on [141], and [Chap. 7](#) on intuitionistic trilattice logics presents material from [273]. The idea of a harmonious many-valued logic, presented in [Chap. 8](#), was first developed in [275]. The application of generalized truth values in the context of the discussion of Suszko's Thesis in [Chap. 9](#) originates from our paper [276]. Moreover, this chapter also includes some basic results about the interconnections between various entailment relations reported in [234]. At this time we would also like to thank Springer-Verlag, Cambridge University Press, and Oxford University Press for their kind permission in letting us make use of the papers in question in this book.

Finally, we would like to express warm gratefulness to our families who have always been for us a constant source of understanding, support, and encouragement. We dedicate this book to our spouses, *Natalia Shramko* and *Petra Wansing*, and thank them for all their love and patience.

Kryvyi Rih and Bochum  
September 2010

Yaroslav Shramko  
Heinrich Wansing



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