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Stochastic Control Theory

Dynamic Programming Principle

Springer

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Preface

The purpose of this book is to provide an introduction to stochastic controls theory, via the method of dynamic programming. The dynamic programming principle, originated by R. Bellman in 1950s, is known as the two stage optimization procedure. When we control the behavior of a stochastic dynamical system in order to optimize some payoff or cost function, which depends on the control inputs to the system, the dynamic programming principle gives a powerful tool to analyze problems. Exploiting the dependence of the value function (optimal payoff) on its terminal cost function, we will construct a nonlinear semigroup which allows one to formulate the dynamic programming principle and whose generator provides the Hamilton–Jacobi–Bellman equation. Here we are mainly concerned with finite time horizon stochastic controls. We also apply the semigroup approach to control-stopping problems and stochastic differential games, and provide with examples from the area of financial market models.

This book is organized as follows. Chapters 1–4 deal with completely observable finite-dimensional controlled diffusions. Chapters 5 and 6 are concerned with Hilbert space valued stochastic processes, related to partially observable control problems.

Chapter 1 is a review of stochastic analysis and stochastic differential equations with random coefficients for later uses. Chapter 2 deals with control problems with finite-time horizon. By a time-discretization method we construct a semigroup, associated with the value function, whose generator provides the Hamilton–Jacobi–Bellman equation. When the value function is smooth, it becomes a classical solution of the Hamilton–Jacobi–Bellman equation. However, it satisfies the equation in viscosity sense even if it is not smooth. Chapter 3 is concerned with viscosity solutions of nonlinear parabolic equation, including Hamilton–Jacobi–Bellman equations of stochastic controls and also stochastic optimal control-stopping problems. Chapter 4 presents zero sum, two-player, time-homogeneous, stochastic differential games and the Isaacs equations. We consider stochastic differential games by using progressive strategies. Then we construct semigroups associated with the upper and lower values, by using a semidiscretization method. These semigroups lead to the formulation of the dynamic programming principle and to the upper and lower Isaacs equations. The link between stochastic control and differential game is given via the risk sensitive control. Chapter 5 is a review on stochastic evolution equations on Hilbert spaces, in particular stochastic parabolic equations with colored Wiener noises. Basic definitions and results and Itô's formula are presented. Chapter 6 is concerned with control problems for Zakai equations. We again construct semigroups associated with the value functions. The dynamic programming principle and viscosity solutions of Hamilton–Jacobi–Bellman equations on Hilbert spaces are treated by using results obtained in the previous chapters. We show the connection between controlled Zakai equations and control of partially observable diffusions.

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Contents

Notations

Let a and b be real numbers.
$a \vee b = \max\{a, b\}, \quad a \wedge b = \min\{a, b\}$
$a^+ = \max\{a, 0\}, \quad a^- = \min\{-a, 0\}$
$\delta_{i,j}$ = Kronecker symbol
$\mathbb{R}^d$ denotes d-dimensional Euclidean space
x^i denotes the i-th coordinate of $x \in \mathbb{R}^d$
$x \cdot y = \sum_{i=1}^d x^i y^i, \quad |x|^2 = x \cdot x$
$\mathbb{R}^d_+ = \{x \in \mathbb{R}^d; \quad x^i > 0, \quad i = 1, \ldots, d\}$
$S_r = \{x \in \mathbb{R}^d; \quad |x| \leq r\}$
$\mathbb{R}^d \otimes \mathbb{R}^m$ denotes the set of $d \times m$ matrices
$A^i_j = (i, j)$ entry of $A \in \mathbb{R}^d \otimes \mathbb{R}^m$
$A^\top$ = transpose of A
tr A = trace of $A \in \mathbb{R}^d \otimes \mathbb{R}^d$
$|A|^2 = \mathrm{tr}(A^\top A) = \mathrm{tr}(AA^\top) \quad$ for $A \in \mathbb{R}^d \otimes \mathbb{R}^m$
I_d = d-dimensional unit matrix
S^d = the set of symmetric $d \times d$ matrices
$S^d_+ = \{A \in S^d; \quad A$ is nonnegative definite$\}$
$S^d_{++} = \{A \in S^d; \quad A$ is positive definite$\}$

$\mathcal{B}(U)$ = the σ-field generated by all open subsets of U
$L^p(\Omega, \mathcal{G}; \mathbb{R}^d)$ = the set of $\mathcal{G}$-measurable d-dimensional random variables with $E \mid \xi \mid^p < \infty, \quad$ for $\quad p \geq 1$
$L^p([0, T] \times \Omega, (\mathcal{F}_t); \mathbb{R}^d)$ = the set of $(\mathcal{F}_t)$-progressively measurable d-dimensional processes with $\int_0^T E \mid X(t) \mid^p \, dt < \infty, \quad$ for $\quad p \geq 1$
$\mathbf{\Gamma} = L^\infty([0, T] \times \Omega, (\mathcal{F}_t); \Gamma)$
$\sigma(\xi)$ = the σ-field generated by ξ
Let Σ be a metric space.
$C(\Sigma)$ = the set of real valued continuous functions defined on Σ
$C_{\mathrm{b}}(\Sigma) = \{\phi \in C(\Sigma); \quad \phi$ is bounded$\}$
$C_{\mathrm{bu}}(\Sigma) = \{\phi \in C_b(\Sigma); \quad \phi$ is uniformly continuous$\}$

$C_{\mathrm{p}}(\Sigma) = \{\phi \in C(\Sigma);\quad \phi$ is polynomial growing$\}$, when Σ is a Banach space.
$C_K^\infty(\mathbb{R}^d) = \{\phi\, l \in C(\mathbb{R}^d);\quad \phi$ has compact support and continuous derivatives of any order$\}$
$C^{12}((0,T)\times\mathbb{R}^d) = \{\phi \in C((0,T)\times\mathbb{R}^d); \partial_t\phi,\ \partial_i\phi,\ \partial_{ij}\phi \in C((0,T)\times\mathbb{R}^d),$
$i, j = 1,\ldots,d\}$,
where $\partial_t = \frac{\partial}{\partial t},\quad \partial_i = \frac{\partial}{\partial x_i},\quad \partial_{ij} = \frac{\partial^2}{\partial x_i \partial x_j}$
$C^{12}([0,T)\times\mathbb{R}^d) = \{\phi \in C([0,T)\times\mathbb{R}^d) \cap C^{12}((0,T)\times\mathbb{R}^d);$
$\partial_t\phi,\ \partial_i\phi,\ \partial_{ij}\phi$ can be extended to continuous functions on
$C([0,T)\times\mathbb{R}^d),\quad i,j = 1,\ldots,d\}$
$C^{12}((0,T]\times\mathbb{R}^d)$ and $C^{12}([0,T]\times\mathbb{R}^d)$ are defined similarly
$\chi_\Lambda =$ the indicator function of the set Λ
$\partial_x\phi =$ gradient vector of ϕ
$\partial_{xx}\phi =$ matrix of second order partial derivatives of $\phi = (\partial_{ij}\phi)_{i,j=1,\ldots,d}$
$\phi/\Lambda = \phi$ restricted to a set Λ

$H_0 = L^2(\mathbb{R}^d)$ with the usual inner product $(\cdot,\cdot)$ and norm $\|\cdot\|$
$\mathbf{I} =$ identity mapping
$\Delta =$ Laplacian operator
$H_p = \{\phi \in H_0;$ generalized derivatives of ϕ up to order p belong to $H_0\},\quad p = 1, 2, \ldots$
$\|\phi\|_{H_p} = \|(\mathbf{I}-\Delta)^{\frac{p}{2}}\phi\|$
$H_{-1} = \{$Borel functions $\phi;\quad (\mathbf{I}-\Delta)^{\frac{-1}{2}}\phi \in H_0\}$
$|||\phi||| = \|\phi\|_{H_1},\quad \|\phi\|_* = \|\phi\|_{H_{-1}} = \|(\mathbf{I}-\Delta)^{\frac{-1}{2}}\phi\|$
$\langle\phi,\psi\rangle =$ duality product between $\phi \in H_{-1}$ and $\psi \in H_1$
Let $\mathbb{H}$ and $\mathbb{Y}$ be Hilbert spaces.
$L^2(\mathbb{H};\ \mathbb{Y}) =$ Hilbert space of Hilbert–Schmidt operators from $\mathbb{H}$ into $\mathbb{Y}$
$\|\Phi\|_Q = \|\Phi Q^{\frac{1}{2}}\|_{L^2(\mathbb{H};\mathbb{Y})}$, where Q is a symmetric and nonnegative definite operator on $\mathbb{H}$

$\mathbb{M}^{2c}([0,T],\ (\mathcal{F}_t);\ \mathbb{H}) =$ set of continuous and square integrable $\mathbb{H}$-valued $(\mathcal{F}_t)$-martingales on $[0,T]$
$\langle M\rangle =$ quadratic variation process of M
$\langle M, N\rangle =$ quadratic variation process corresponding to M and N
$D^pF(z) = p$-th Fréchet derivative of F at z
$C^{12}([0,T]\times\mathbb{H}) = \{F \in C([0,T]\times\mathbb{H});\quad \partial_tF,\ DF$ and D^2F are continuous on $[0,T]\times\mathbb{H}\}$
$A \Rightarrow B =$ if A then B

Abbreviations

RHS = right-hand side
LHS = left-hand side
USC = upper semicontinuous
LSC = lower semicontinuous
ONB = orthonormal basis
w.r.t. = with respect to
a.e. = almost everywhere