Part II

Filling Functions

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Notation

$\preceq,\ \succeq,\ \simeq$	$f, g : [0,\infty) \to [0,\infty)$ satisfy $f \preceq g$ when there exists $C > 0$ such that $f(n) \leq Cg(Cn+C)+Cn+C$ for all n, satisfy $f \succeq g$ when $g \preceq f$, and satisfy $f \simeq g$ when $f \preceq g$ and $g \preceq f$. These relations are extended to functions $f : \mathbb{N} \to \mathbb{N}$ by considering such f to be constant on the intervals $[n, n+1)$.
$a^b,\ a^{-b},\ [a,b]$	$b^{-1}ab,\ b^{-1}a^{-1}b,\ a^{-1}b^{-1}ab$
$Cay^1(G,X)$	the Cayley graph of G with respect to a generating set X
$Cay^2(\mathcal{P})$	the Cayley 2-complex of a presentation $\mathcal{P}$
$\mathbb{D}^n$	the n-disc $\left\{(x_1,\ldots,x_n) \in \mathbb{R}^n \mid \sum_{i=1}^n {x_i}^2 \leq 1\right\}$
$\mathrm{Diam}(\Gamma)$	$\max\{\,\rho(a,b) \mid \text{vertices } a, b \text{ in } \Gamma\}$, where ρ is the combinatorial metric on a finite connected graph Γ
d_X	the word metric with respect to a generating set X
$\ell(w)$	word length; i.e. the number of letters in the word w
$\ell(\partial\Delta)$	the length of the boundary circuit of Δ
$\mathbb{N}$, $\mathbb{R}$, $\mathbb{Z}$	the natural numbers, real numbers, and integers
R^{-1}	$\left\{r^{-1} \mid r \in R\right\}$, the inverses of the words in R
$\mathbb{S}^n$	the n-sphere, $\left\{(x_1,\ldots,x^{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1} = 1\right\}$
$\mathrm{Star}(\Delta_0)$	for a subcomplex $\Delta_0 \subseteq \Delta$, the union of all closed cells in Δ that have non-empty intersection with Δ_0
(T, T^*)	a dual pair of spanning trees — see Section 1.2
w^{-1}	the inverse ${x_n}^{-\varepsilon_n}\ldots{x_2}^{-\varepsilon_2}{x_1}^{-\varepsilon_1}$ of a word $w = {x_1}^{\varepsilon_1}{x_2}^{\varepsilon_2}\ldots{x_n}^{\varepsilon_n}$.
$\langle X \mid R\rangle$	the presentation with generators X and defining relations (or *relators*) R
X^{-1}	$\left\{x^{-1} \mid x \in X\right\}$, the formal inverses x^{-1} of letters x in an alphabet X
$(X \cup X^{-1})^*$	the free monoid (i.e. the words) on $X \cup X^{-1}$
Δ	a van Kampen diagram
ε	the empty word

Diagram measurements

$\mathrm{Area}(\Delta)$	the number of 2-cells in Δ
$\mathrm{DGL}(\Delta)$	$\min\left\{\ \mathrm{Diam}(T) + \mathrm{Diam}(T^*)\ \middle|\ T \text{ a spanning tree in } \Delta^{(1)}\ \right\}$
$\mathrm{EDiam}(\Delta)$	the diameter of Δ as measured in the Cayley 2-complex
$\mathrm{FL}(\Delta)$	the filling length of Δ — see Section 1.2
$\mathrm{GL}(\Delta)$	the diameter of the 1-skeleton of the dual of Δ
$\mathrm{IDiam}(\Delta)$	the diameter of the 1-skeleton of Δ
$\mathrm{Rad}(\Delta)$	$\max\{\ \rho(a,\partial\Delta)\ \mid\ \text{vertices } a \text{ of } \Delta\ \}$ as measured in $\Delta^{(1)}$

Filling functions — see Section 1.2 unless otherwise indicated

$\mathrm{Area} : \mathbb{N} \to \mathbb{N}$	the Dehn function
$\mathrm{DGL} : \mathbb{N} \to \mathbb{N}$	the simultaneous diameter and gallery length function
$\mathrm{EDiam} : \mathbb{N} \to \mathbb{N}$	the extrinsic diameter function
$\mathrm{FL} : \mathbb{N} \to \mathbb{N}$	the filling length function
$\mathrm{GL} : \mathbb{N} \to \mathbb{N}$	the gallery length function
$\mathrm{IDiam} : \mathbb{N} \to \mathbb{N}$	the intrinsic diameter function
$\overline{\mathrm{IDiam}} : \mathbb{N} \to \mathbb{N}$	the upper intrinsic diameter function — see Section 2.1
$\mathrm{Rad} : \mathbb{N} \to \mathbb{N}$	the radius function — see Section 4.2
$\overline{\mathrm{Rad}} : \mathbb{N} \to \mathbb{N}$	the upper radius function — see Section 4.2

Introduction

The Word Problem was posed by Dehn [32] in 1912. He asked, given a group, for a systematic method (in modern terms, an *algorithm*) which, given a finite list (a *word*) of basic group elements (*generators* and their formal inverses), declares whether or not their product is the identity. One of the great achievements of 20^{th} century mathematics was the construction by Boone [13] and Novikov [73] of finitely presentable groups for which no such algorithm can exist. However, the Word Problem transcends its origins in group theory and rises from defeat at the hands of decidability and complexity theory, to form a bridge to geometry — to the world of isoperimetry and curvature, local and large-scale invariants, as brought to light most vividly by Gromov [61].

So where does geometry enter? Groups act: given a group, one seeks a space on which it acts in as favourable a manner as possible, so that the group can be regarded as a discrete approximation to the space. Then a dialogue between geometry and algebra begins. And where can we find a reliable source of such spaces? Well, assume we have a finite generating set X for our group G. (All the groups in this study will be finitely generated.) For $x, y \in G$, define the distance $d_X(x, y)$ in the *word metric* d_X to be the length of the shortest word in the generators and their formal inverses that represents $x^{-1}y$ in G. Then

$$d_X(zx, zy) = d_X(x, y)$$

for all $x, y, z \in G$, and so left multiplication is action of G on (G, d_X) by isometries.

However (G, d_X) is discrete and so appears geometrically emaciated ("boring and uneventful to a geometer's eye" in the words of Gromov [61]). Inserting a directed edge labelled by a from x to y whenever $a \in X$ and $xa = y$ gives a skeletal structure known as the *Cayley graph* $Cay^1(G, X)$. If G is given by a finite presentation $\mathcal{P} = \langle X \mid R \rangle$ we can go further: attach flesh to the dry bones of $Cay^1(G, X)$ in the form of 2-cells, their boundary circuits glued along edge-loops around which read words in R. The result is a simply connected space $Cay^2(\mathcal{P})$ on which G acts *geometrically* (that is, properly, discontinuously and cocompactly) that is known as the *Cayley 2-complex* — see Section 1.1 for a precise definition. Further enhancements may be possible. For example, one could seek to attach cells of dimension 3 or above to kill off higher homotopy groups or so that the complex enjoys curvature conditions such as the CAT(0) property (see [21] or Part III of this volume).

Not content with the combinatorial world of complexes, one might seek smooth models for G. For example, one could realise G as the fundamental group of a closed manifold M, and then G would act geometrically on the universal cover $\widetilde{M}$. (If G is finitely presentable then M can be taken to be four dimensional — see [18, A.3].) Wilder non-discrete spaces, *asymptotic cones*, arise from viewing (G, d_X) from increasingly distant vantage points (*i.e.* scaling the metric to d_X/s_n for some sequence of reals with $s_n \to \infty$) and recording recurring patterns using the magic of a non-principal ultrafilter. Asymptotic cones discard all small-scale features of (G, d_X); they are the subject of Chapter 4.

Filling functions, the subject of this study, capture features of discs spanning loops in spaces. The best known is the classical *isoperimetric* function for Euclidean space $\mathbb{E}^m$ — any loop of length ℓ can be filled with a disc of area at most a constant times ℓ^2. To hint at how filling functions enter the world of discrete groups we mention a related algebraic result concerning the group $\mathbb{Z}^m$, the integer lattice in m-dimensional Euclidean space, generated by $x_1, \ldots, x_m$. If w is a word of length n on $\{{x_1}^{\pm 1}, \ldots, {x_m}^{\pm 1}\}$ and w represents the identity in $\mathbb{Z}^m$ then, by cancelling off pairs $x_i {x_i}^{-1}$ and ${x_i}^{-1} x_i$, and by interchanging adjacent letters at most n^2 times, w can be reduced to the empty word.

This qualitative agreement between the number of times the commutator relations $x_i x_j = x_j x_i$ are applied and the area of fillings is no coincidence; such a relationship holds for all finitely presented groups, as will be spelt out in Theorem 1.6.1 (*The Filling Theorem*). The bridge between continuous maps of discs filling loops in spaces and this computational analysis of reducing words is provided by *van Kampen* (or *Dehn*) *diagrams*. The Cayley 2-complex of the presentation

$$\mathcal{P} \ := \ \langle x_1, \ldots, x_m \mid [x_i, x_j], \forall i, j \in \{1, \ldots, m\} \,\rangle$$

of $\mathbb{Z}^m$ is the 2-skeleton of the standard decomposition of $\mathbb{E}^m$ into an infinite array of m-dimensional unit cubes. A word w that represents 1 in $\mathcal{P}$ (or, indeed, in any finite presentation $\mathcal{P}$) corresponds to an edge-loop in $\mathit{Cay}^2(\mathcal{P})$. As Cayley 2-complexes are simply connected such edge-loops can be spanned by filling discs and, in this combinatorial setting, it is possible and appropriate to take these homotopy discs to be combinatorial maps of planar 2-complexes homeomorphic to (possibly *singular*) 2-discs into $\mathit{Cay}^2(\mathcal{P})$. A *van Kampen diagram* for w is a graphical demonstration of how it is a consequence of the relations R that w represents 1; Figure 1.4 is an example. So the Word Problem amounts to determining whether or not a word admits a van Kampen diagram. (See *van Kampen's Lemma: Lemma* 1.4.1.)

Filling functions for finite presentations of groups (defined in Chapter 1) record geometric features of van Kampen diagrams. The best known is the *Dehn function* (or *minimal isoperimetric function*) of Madlener & Otto [68] and Gersten [46]; it concerns *area* — that is, number of 2-cells. In the example of $\mathbb{Z}^m$

this equates to the number of times commutator relations have to be applied to reduce w to the empty word — in this sense (that is, *in the Dehn proof system* — see Section 1.5) the Dehn function can also be understood as a non-deterministic TIME complexity measure of the Word Problem for $\mathcal{P}$ — see Section 1.5. The corresponding SPACE complexity measure is called the *filling length function* of Gromov [61]. It has a geometric interpretation — the filling length of a loop γ is the infimal length L such that γ can be contracted down to its base vertex through loops of length at most L. Other filling functions we will encounter include the *gallery length*, and *intrinsic* and *extrinsic diameter functions*. All are group invariants in that whilst they are defined with respect to specific finite presentations, their qualitative growth depends only on the underlying group; moreover, they are quasi-isometry invariants, that is, qualitatively they depend only on the large-scale geometry of the group — see Section 1.7 for details.

In Chapter 2 we examine the interplay between different filling functions — this topic bares some analogy with the relationships that exist between different algorithmic complexity measures and, as with that field, many open questions remain. The example of nilpotent groups discussed in Chapter 3 testifies to the value of simultaneously studying multiple filling functions. Finally, in Chapter 4, we discuss how the geometry and topology of the asymptotic cones of a group G relates to the filling functions of G.

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