Part I

Dehn Functions and Non-Positive Curvature

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Preface

In this portion of the course we shall explore some ways of constructing groups with specific Dehn functions, and we shall look at connections between Dehn functions and non-positive curvature. The presentation of the material will proceed via a series of concrete examples. Further, each section contains exercises.

Relevant background topics from the topology of groups (such as *graphs of groups* and *graphs of spaces*), and from non-positively curved geometry (such as *CAT(0) spaces* and *CAT(0) groups*, and *hyperbolic groups*) are introduced with a view to the immediate applications in this course. So we shall learn definitions and statements of the major results in these areas, and proceed to examples and applications rather than spending time on proofs. Here is an outline of how we shall proceed.

First, we study the *snowflake construction*, which produces groups with Dehn functions of the form x^α for a dense set of exponents $\alpha \geqslant 2$, including all rationals. These groups and constructions are far from the non-positively curved universe; for instance, the snowflakes are not even subgroups of the non-positively curved groups mentioned in the next paragraph.

The next series of examples are all subgroups of non-positively curved groups. The non-positively curved groups in question are *CAT(0) groups* and *hyperbolic groups*. Subgroups of non-positively curved groups are not well understood at present; the collection of subgroups is potentially a vast reservoir of new geometries and groups. One key difficulty in this field is that there is a real dearth of concrete examples. Another problem is that there are very few good tools for analyzing the geometry of subgroups of non-positively curved groups.

We begin by examining a construction for embedding certain amalgamated doubles of groups into non-positively curved groups that has its foundations in a paper of Bieri. As an application, we construct a family of CAT(0) 3-dimensional cubical groups which contain subgroups with Dehn functions of the form x^n for each $n \geqslant 3$. The groups that are being doubled are free-by-cyclic groups which are the fundamental groups of non-positively curved squared complexes. We define *Morse functions* on affine cell complexes, and use Morse theoretic techniques to see that the fundamental groups of the squared complexes above are indeed free-by-cyclic.

The Morse theory techniques are applied to non-positively curved cubical complexes for the remaining applications and examples. In one application we look at Morse functions on cubical complexes corresponding to *right-angled Artin groups*. The Artin group is the fundamental group of the associated cubical complex, and the circle-valued Morse function induces an epimorphism from the Artin group to $\mathbb{Z}$. The geometry of the kernel of this epimorphism is intimately related to the geometry and topology of the *level sets* of a lift of the Morse function to the universal cover. As examples, we produce right-angled Artin groups containing subgroups which have Dehn function of the form x^n for $n \geqslant 3$. These examples have a very different feel to the embedded doubled examples above. In the doubled examples, the Dehn function exponent is closely related to the *distortion* of free subgroups in the doubled group. This is not the case with the right-angled Artin examples.

As a final example, we construct a branched cover of a 3-dimensional cubical complex, with the following properties. The fundamental group is hyperbolic. There is an epimorphism to $\mathbb{Z}$ whose kernel is finitely presented but not hyperbolic. The kernel is known not to be hyperbolic because it is not of type F_3; an explicit calculation of its Dehn function is yet to be carried out.

Morse theory is the major background theme in this portion of the course. It is used explicitly in the later sections on Artin groups and on branched covers. It is used to recognize free-by-cyclic groups in the section on embedding doubles. It is also the motivation for the *torus construction* which produces the vertex groups in the graph of groups description of the snowflake groups. The torus construction leads to a whole range of groups with interesting geometry and topology. These include a famous example due to Stallings of a finitely presented group which is not of type F_3. The torus construction leads to quick descriptions for a range of variations of Stallings' example, some of which have cubic Dehn functions. Some may have a quadratic Dehn function. There is much to explore here.

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