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S.A. Grigoryan
T.V. Tonev

Shift-invariant Uniform Algebras on Groups

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Authors:

Suren A. Grigoryan
Chebotarev Institute for
Mathematics and Mechanics
Kazan State University
Universitetskaya 17
Kazan 420008, Tatarstan
Russia
e-mail: Suren.Grigorian@ksu.ru

Thomas V. Tonev
Department of Mathematical Sciences
University of Montana
Missoula, MT 59812-0864
USA
e-mail: TonevTV@mso.umt.edu

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Preface

Shift-invariant algebras are uniform algebras of continuous functions defined on compact connected groups, that are invariant under shifts by group elements. They are outgrowths of generalized analytic functions, introduced almost fifty years ago by Arens and Singer, and are the central object of this book. Associated algebras of almost periodic functions of real variables and of bounded analytic functions on the unit disc are also considered and carried along within the shift-invariant framework. The adopted general approach leads to non-standard perspectives, never-asked-before questions, and unexpected properties.

The book is based mainly on our quite recent, some even unpublished, results. Most of its basic notions and ideas originate in [T2]. Their further development, however, can be found in journal or preprint form only.

Basic terminology and standard properties of uniform algebras are presented in Chapter 1. Associated algebras, such as Bourgain algebras, polynomial extensions, and inductive limit algebras are introduced and discussed. At the end of the chapter we present recently found conditions for a mapping between uniform algebras to be an algebraic isomorphism. In Chapter 2 we give fundamentals, various descriptions and standard properties of three classical families of functions – almost periodic functions of real variables, harmonic functions, and H^p -functions on the unit circle. Later on, in Chapter 7, we return to some of these families and extend them to arbitrary compact groups. Chapter 3 is a survey of basic properties of topological groups, their characters, dual groups, functions and measures on them. We introduce also the instrumental for the sequel notion of weak and strong hull of a semigroup.

Chapter 4 is devoted to shift-invariant algebras. We describe the spaces of automorphisms and of peak subgroups of shift-invariant algebras, and show that the algebraic properties of the generating semigroup S have a significant impact on the properties of the associated shift-invariant algebra A_S . For example, whether analogues of the classical Radó's theorem for null-sets of analytic functions, and of Riemann's theorem for removable singularities hold in a shift-invariant algebra A_S depends on specific algebraic properties of the generating semigroup S . Asymptotically almost periodic functions on \mathbb{R} , which share many properties with almost periodic functions, are introduced at the end of the chapter. Extendability of linear multiplicative functionals from smaller to larger shift-invariant algebras is the focal point of Chapter 5. The subject is naturally related with the extendability of non-negative semicharacters from smaller to larger semigroups and, equivalently, of their logarithms, called also additive weights. We give necessary and sufficient conditions for extendability of individual weights, as well as of the entire family of weights on a semigroup. These conditions imply various corona-type theorems. For instance, if S is a semigroup of \mathbb{R} containing the origin, then the algebra of almost periodic functions in one real variable with spectrum in S does not have a \mathbb{C}_+ -corona if and only if all non-negative semicharacters on S are monotone

decreasing, or equivalently, if and only if the strong hull of S coincides with the positive half of the group envelope of S . On the other hand, the same conditions imply necessary and sufficient conditions for the related subalgebra of bounded analytic functions on the unit disc \mathbb{D} to possess a \mathbb{C}_+ -corona and a \mathbb{D} -corona. In Chapter 6 we discuss big disc algebras of generalized analytic functions on a compact abelian group G , an important class of shift-invariant algebras, also known as G -disc algebras. We describe their Bourgain algebras, orthogonal measures and primary ideals.

In Chapter 7 we extend the notion of harmonic and H^p -functions to compact abelian groups, and present corresponding Fatou-type theorems. In Chapter 8 we utilize inductive limits of classical algebras to study and generalize shift-invariant algebras on G -discs. In particular, we show that any sequence Φ of inner functions on the unit disc generates an inductive limit algebra, $H^\infty(\mathcal{D}_\Phi)$, of so called Φ -hyper-analytic functions on the associated big disc \mathcal{D}_Φ . They are generalizations of hyper-analytic functions from [T], and similarly to them do not have a G -disc-corona, i.e. there exists a standard dense embedding of the big disc \mathcal{D}_Φ into the maximal ideal space of $H^\infty(\mathcal{D}_\Phi)$. We introduce also the class of Blaschke algebras, which are inductive limits of sequences of disc algebras connected with finite Blaschke products.

The selection of topics depended entirely on our own research interests. Many other related topics could not be included, or even mentioned. All chapters are provided with historical notes, references, brief remarks, comments, and unsolved problems. We do not necessarily claim credit for any uncited result. It may be an immediate consequence of previous assertions, or, part of the common mathematical knowledge, or, may have a history difficult to be traced.

The book is addressed primarily to those interested in analytic functions and commutative Banach algebras, though it could be useful to a wide range of research mathematicians and graduate students, familiar only with the fundamentals of complex and functional analysis.

Over the years our thinking in the area has been stimulated and encouraged by discussions and communication with several experts, among which we would like to mention Hugo Arizmendi, Richard Aron, Andrew Browder, Joseph Cima, Brian Cole, Joseph Diestel, Evgeniy Gorin, Farhad Jafari, Krzysztof Jarosz, Paul Muhly, Rao Nagisetty, Scott Saccone, Sadahiro Saeki, Anatoly Sherstnev, Andrzej Sołtysiak, Edgar Lee Stout, John Wermer, and Wiesław Żelazko. Special thanks are due to the participants – current and former – of the Analysis seminar at the University of Montana: Gregory St.George, Karel Stroethoff, Elena Toneva, George Votruba, and Keith Yale for their encouragement and support. We also mention with pleasure and gratitude the contribution of our students Tatyana Ponkrateva from Kazan State University, Aaron Luttmann and John Case from the University of Montana, and especially Scott Lambert, who read the entire text and suggested many improvements.

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Missoula, Montana

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